The calculation of the CE with the wave function (3.1) can be carried out by the formulas given in reference 5, when a charge distribution of Yukawa type is assumed. The factor $e^{-\beta_i D}$ in the formula for potential energy[7] should be deleted. If an exponential charge distribution is assumed, the calculation becomes much more complicated.

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PHYSICAL REVIEW VOLUME 128, NUMBER 2 OCTOBER 15, 1962

Distorted-Waves Theory of Double-Excitation by Inelastic Scattering*

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(Received June 1, 1962)

It is shown how the distorted-waves theory of inelastic scattering can give rise to angular distributions with "anomalous phase," such as observed in recent experiments with alpha particles. These occur when the scattering takes place through second-order effects; excitation of a two-phonon vibrational state is studied explicitly. There is an important cancellation between the amplitude for simultaneous excitation and part of that for successive excitation, without which the anomalous phase would not be observed. These conclusions are contrasted with the predictions of a theory using plane-wave Born approximation. Further study of the cancellation is suggested as a sensitive test of the optical model.

I. INTRODUCTION

THE inelastic scattering of strongly-absorbed projectiles is known[1,2] to show a clearly-defined phase-rule relationship among the oscillatory angular distributions for exciting different states of a given target. According to this rule, angular distributions corresponding to odd values of the angular momentum transfer L have their maxima and minima out of phase with those of angular distributions corresponding to even L. Also the odd-L patterns are in phase with the elastic angular distributions. The conditions under which these rules should be reliable are not very restrictive,[3,4] and are fulfilled very well by (α,α') reactions at energies of about 40 MeV. The use of such reactions in the excitation of the lowest 2^+ and 3^- levels has provided many experimental verifications of the phase rule.

It was especially interesting, therefore, when several (α,α') excitations of known 4^+ levels of even nuclei were found[5,6] to be in phase with the 3^- excitations of the same nuclei, and out of phase with the 2^+ excitations. The 4^+ levels in question are believed to be part of the two-phonon triplet of quadrupole vibrational states of these nuclei, and it was suggested that two-phonon excitation somehow reverses the phase rule.

Indeed Blair[7] and Drozdov[8] applied a formula of the adiabatic theory, for the scattering from a "black" ellipsoid, and discovered that in this model the part of the scattering amplitude which is second order in the deformation does give agreement with experiment. The

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second-order terms which are appreciable at large angles have oscillations with the reversed phase, and their envelope falls more slowly with angle. Both properties agree with experiment. An extended and improved application of the adiabatic theory has been developed by Austern and Blair[9] and gives the same results.

However, the adiabatic theory does not distinguish between *simultaneous* and *successive* excitation of two phonons, so it does not give a complete picture of the reaction mechanism. These two types of excitation are defined if one considers a Taylor expansion of the optical potential of a nucleus whose surface has a 2^k-pole deformation, defined by

$$R(\Omega)=R_0[1+\textstyle\sum_q \alpha_q Y_k{}^q(\Omega)].$$

If we assume that the potential U depends only on the distance from this surface, we may write

$$\begin{aligned} U(r-R)&=U(r-R_0)+\Delta U,\\ \Delta U&=-R_0[\textstyle\sum_q \alpha_q Y_k{}^q(\Omega)]U'\\ &\quad+\tfrac{1}{2}R_0{}^2[\textstyle\sum_q \alpha_q Y_k{}^q(\Omega)]^2U''+\cdots. \end{aligned} \quad (1)$$

In the vibrational model the deformation parameter, α_q, creates or annihilates a phonon[10] with angular momentum k and z component q. The U' and U'' are derivatives of U with respect to r, evaluated at $\alpha=0$. Then $U(r-R_0)$ is interpreted as the spherical potential giving rise to elastic scattering, while the U' term is able to excite a single phonon, and so forth. *Simultaneous* excitation of two phonons then refers to the direct (or crossover) transition (from 0^+ to 4^+, say, if $k=2$) by the operation of the nonlinear term of ΔU, that proportional to U''. *Successive* excitation refers to the two-step transition (0^+ to 2^+ to 4^+, if $k=2$) by repeated operation of the U' term. The adiabatic theory treats only the combined result of these two terms. It is interesting to study the reaction mechanism in greater detail, not only to understand the role the optical potential plays in giving these two terms, but also to understand the conditions under which the results of the adiabatic theory are obtained. A perturbative approach will permit such a more detailed analysis.

A perturbation theory for two-phonon excitation was given by Lemmer, de-Shalit, and Wall.[11] They simplified their considerations by using plane-wave Born approximation, and then found that the U'' term of (1) dominated the process, so that the two phonons were excited simultaneously. They obtained both the anomalous phase relationship and the slow dropoff of the envelope of the angular distribution, and also could predict the magnitude of the 4^+ cross section relative to that of the 2^+ cross section.

Nevertheless, plane-wave Born approximation is certainly a false theory for (α,α') reactions. In particular, as we shall show below, its predictions for the second-order process are very misleading. Fortunately, distorted-waves Born approximation (DWB) describes these just as accurately as it does nearly all other direct reactions, and may be applied instead.

Normally DWB only is carried to first Born approximation, that is, to first order in ΔU, and in this way includes only one of the two terms which are second order in the deformation, namely the U'' term. To include *successive* excitation by the U' term it is necessary to go to second Born approximation. In this way, a series in powers of the deformation is generated. It is interesting to recall that practical application of the adiabatic theory also requires the use of a series in powers of the deformation. To first order in the deformation the two theories are identical in the limit of zero energy loss, and the adiabatic theory may be thought of as a heuristic reformulation of DWB. Now this same correspondence is followed to second order. These two theories have the same content, but use different techniques, such that one theory is simpler, but the other can be more accurate. Different kinds of insights thereby are obtained.

It will be seen in the present paper that to obtain agreement with experiment both terms of Eq. (1) must be carried, and that their ratio probably is essentially as predicted by the simple picture which Eq. (1) gives for the optical potential of a deformed nucleus. Because the analysis is thus far only semiquantitative it is not clear how good a fit to experiment the nuclear model embodied in Eq. (1) as it stands can give. However, the structure of the double-excitation theory does show that agreement with experiment can be achieved in only one way, and a study of such numbers as are available then shows it to be plausible that this way is successful. What happens is that either term of (1) taken alone gives major disagreement with experiment, contrary to the conclusions of reference 11, but that the two terms combine so that the parts which disagree with experiment cancel. Numerical calculations of improved accuracy may be used to study this cancellation, and it may be that new information about deformed optical potentials and the nature of collective excited states can thereby be derived.[12] It is interesting that the one part of the transition amplitude which survives the cancellation, and which does have the right properties to agree with experiment, comes from the *successive* excitation. This result reverses the order of importance which had been found for the two terms by the plane wave analysis, and is one of the most

[9] N. Austern and J. S. Blair, (to be published). Several approximations from this article are employed in the present paper.

[10] A. Bohr and B. R. Mottelson, Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd. **27**, No. 16 (1953).

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[12] Such calculations have been begun by Brian Buck, Phys. Rev. **127**, 940 (1962).

striking contrasts yet seen between DWB and plane wave calculations.[13]

II. CALCULATION OF THE CROSS SECTION

The transition amplitude for exciting an even nucleus to a state with spin I, z component M, is

$$T_{fi}{}^{M}=\langle v_I{}^{M}\chi_f{}^{(-)}|\,\Delta U\,|\Psi_i{}^{(+)}\rangle.$$

To get all the second-order terms in T_{fi} it is necessary to carry the total wave function $\Psi_i{}^{(+)}$ to first order:

$$\Psi_i{}^{(+)}\approx[1+(E+i\epsilon-H_0)^{-1}\Delta U]v_0\chi_i{}^{(+)}.$$

Then the amplitude for double-excitation is the sum of the second-order terms and is

$$T_{fi}{}^{M}=R_0{}^{2}\langle v_I{}^{M}\chi_f{}^{(-)}|\,(\alpha Y)U'(E+i\epsilon-H_0)^{-1}(\alpha Y)U' +\tfrac{1}{2}(\alpha Y)^{2}U''|\,v_0\chi_i{}^{(+)}\rangle, \quad (2)$$

where (αY) is an abbreviation for $\sum_q \alpha_q Y_k{}^{q}(\Omega)$. Some other notation used above is that $\chi_i{}^{(+)}$ and $\chi_f{}^{(-)}$ are the initial state and final state distorted waves, satisfying the usual boundary conditions,[2] and that v_0 and $v_I{}^{M}$ are the eigenstates of the nuclear surface oscillation. It will be convenient to ignore the energy differences among the vibrational eigenstates; these only give small corrections at the kinetic energies in which we are interested. The distorted wave functions which will be used are

$$\chi_i{}^{(+)}=[(4\pi)^{\frac{1}{2}}/k_0r]\sum_l i^{l}(2l+1)^{\frac{1}{2}}e^{i\sigma_l}f_l(k_0,r)Y_l{}^{0}(\Omega), \quad (3a)$$

$$\chi_f{}^{(-)*}=[4\pi/k_0r]\sum_{lm} i^{-l}e^{i\sigma_l}f_l(k_0,r) \times Y_l{}^{m}(\Theta,0)Y_l{}^{m*}(\Omega). \quad (3b)$$

Here, f_l is the radial wave function corresponding to angular momentum $l\hbar$, Θ is the scattering angle, and k_0 is the momentum of the incoming beam, assumed above to equal that of the outgoing beam. Further details of notation may be found in reference 2, and also in the Appendix of the present article.

The first term of Eq. (2) is the term which describes successive excitation of two phonons. It involves as intermediate states not only the one-phonon state of the target nucleus, but also the complete set of continuum states of the projectile. Fortunately, it is not necessary to consider the one-phonon nuclear state explicitly. It is the only state to which α_q can couple the ground state, so quantum-mechanical completeness is fulfilled automatically for the target nucleus coordinates, and we just obtain in this term a product of the operators α_q and $\alpha_{q'}$, the same product as in the second term of Eq. (2). The equation may then be rewritten in a form which greatly facilitates comparison of the two terms:

$$T_{fi}{}^{M}=R_0{}^{2}\sum_{qq'}\langle v_I{}^{M}|\alpha_q\alpha_{q'}|\,v_0\rangle \times\langle\chi_f{}^{(-)}|\,Y_k{}^{q}U'(E+i\epsilon-H_0)^{-1}Y_k{}^{q'}U' +\tfrac{1}{2}Y_k{}^{q}Y_k{}^{q'}U''|\chi_i{}^{(+)}\rangle. \quad (4)$$

Because the spherical harmonic factors in the two terms of Eq. (4) do not have quite the same structure, the q, q' summation still is left in an awkward form. Thus, $Y_k{}^{q}$ and $Y_k{}^{q'}$ have the same arguments in the second term of Eq. (4), but have different arguments in the first term, because in that term they stand on opposite sides of the Green's function. However, it is a very good approximation to treat these factors as commuting with the Green's function, and the simplification thus obtained will now be introduced. It will be justified later, when a detailed analysis of the first term is given. (This same approximation is used in the article of Austern and Blair, reference 9.) Upon combining the spherical harmonics, Eq. (4) becomes

$$T_{fi}{}^{M}\approx\tfrac{1}{2}\beta^{2}R_0{}^{2}C(\mathcal{T}_1{}^{M}+\mathcal{T}_2{}^{M}), \quad (5a)$$

where

$$\beta^{2}CY_I{}^{M*}\equiv\sum_{qq'}\langle v_I{}^{M}|\alpha_q\alpha_{q'}|\,v_0\rangle Y_k{}^{q}Y_k{}^{q'}, \quad (5b)$$

$$\mathcal{T}_1{}^{M}=\langle\chi_f{}^{(-)}|\,Y_I{}^{M*}U''|\,\chi_i{}^{(+)}\rangle, \quad (5c)$$

$$\mathcal{T}_2{}^{M}=\langle\chi_f{}^{(-)}|\,Y_I{}^{M*}U'(E+i\epsilon-H_0)^{-1}U' +U'(E+i\epsilon-H_0)^{-1}U'Y_I{}^{M*}|\,\chi_i{}^{(+)}\rangle. \quad (5d)$$

In Eqs. (5), the parameter β^2 is the mean square deformation in the ground state,

$$\beta^{2}=\langle\textstyle\sum_q|\alpha_q|^{2}\rangle=(2k+1)\hbar\omega_k/2C_k. \quad (6)$$

Also in $\mathcal{T}_2$, the two-step term, the approximate grouping of spherical harmonic factors has been symmetrized, to preserve Hermiticity.

The term $\mathcal{T}_1$ is of the sort treated in usual first-order DWB calculations. It may be evaluated, to give

$$\mathcal{T}_1{}^{M}=(4\pi/k_0{}^{2})\sum_{ll'}i^{l-l'}(2l+1) \times[(2I+1)/(2l'+1)]^{\frac{1}{2}}e^{i(\sigma_l+\sigma_{l'})} \times Y_{l'}{}^{M}(\Theta,0)\langle lI0M|\,l'M\rangle\langle lI00|\,l'0\rangle W_{ll'}, \quad (7a)$$

$$W_{ll'}\equiv\int_0^{\infty}f_{l'}f_lU''dr. \quad (7b)$$

The term $\mathcal{T}_1$ has been studied numerically, using DWB codes developed for the Oak Ridge IBM 7090 computer.[14] It had been hoped that this term would alone be able to give agreement with experiment, as suggested in reference 11. What was discovered instead was that $\mathcal{T}_1$ satisfied with great accuracy the usual phase rule relations for angular momentum I, as shown in Fig. 1. This effect was traced to the radial integrals $W_{ll'}$. These integrals tend to be dominated by such large

[13] The apparent agreement with experiment found in reference 11 resulted from their use of plane-wave Born approximation. A further accident of their use of a square-well potential, is that the details of their calculations closely resemble the corresponding ones of DWB (private communication, N. S. Wall).

[14] R. H. Bassel, R. M. Drisko and G. R. Satchler, Oak Ridge National Laboratory Report ORNL-3240, 1962 (unpublished).

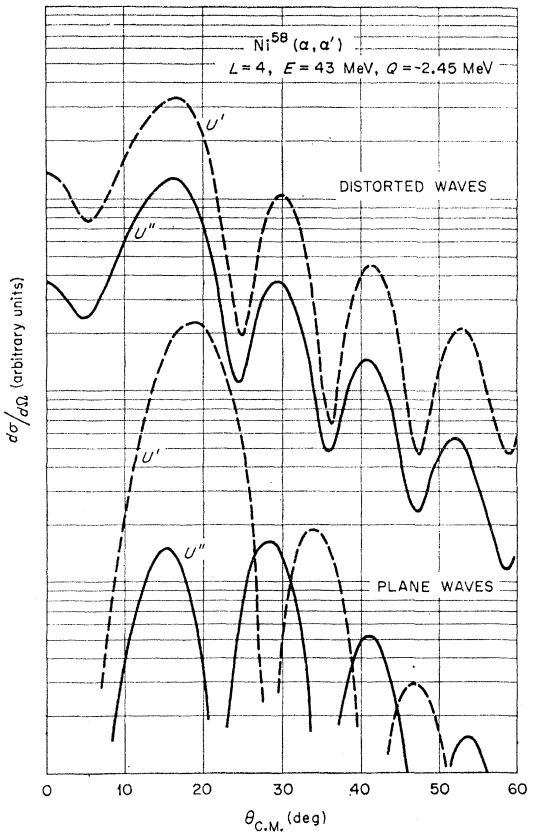

FIG. 1. Angular distribution of 43-MeV alpha particles from Ni^{58} with excitation of the 2.45-MeV 4^+ level. The upper curves were calculated using distorted waves, the lower curves using plane waves. The first derivative of the optical potential was used as interaction for the two curves labeled U', and the second derivative for those labeled U''. A Saxon form was used for the optical potential U, with parameters which fit the elastic scattering, $V=-48$ MeV, $W=-14$ MeV, $R_0=6.14$ F and $a=0.55$ F.

r values that U is essentially exponential over the important region of integration. Then $U''\approx -U'/a$ over this region, and

$$W_{ll'}\approx -a^{-1}\int_0^\infty f_{l'}f_l U' dr, \tag{8}$$

where a gives the rate of falloff of the exponential optical potential. In this circumstance, $\mathcal{T}_1$ has the same angular momentum structure as in usual one-phonon excitation, so it is dominated by a few terms with l and l' near $l_0=k_0R_0$, where R_0 is the nuclear radius. One must, indeed, expect to get the usual phase rule.

The dominance in $\mathcal{T}_1$ of the outer region of r is due to the distortion effects, particularly the strong absorption.[2-4] Figure 2 shows the results of a series of numerical calculations in which the optical potential and Coulomb field from which this distortion arises were progressively reduced to zero. The angular distribution for the $L=4$ excitation by $\mathcal{T}_1$ is compared with several more familiar ones for the one-phonon excitation of $L=2$, 3, 4 levels. We see how distortion reverses the predictions which the plane wave theory makes for the phase of the $L=4$ two-phonon angular distribution, relative to the phases of the usual one-phonon transitions.

The term $\mathcal{T}_2$ is more complicated to treat. Even in the much-simplified Eq. (5d) it is still necessary to introduce an explicit form for the Green's function, and, within this function, to sum over the complete set of continuum states of the projectile. For the present article, it is very helpful to use a bilinear form for the Green s function,

$$(E+i\epsilon-H_0)^{-1}$$
$$\approx -\frac{2}{\pi}\int_0^\infty dk\,\sum_{\lambda\nu}\frac{f_\lambda(k,r)f_\lambda(k,r')Y_\lambda^\nu(\Omega)Y_\lambda^{\nu*}(\Omega')}{rr'\eta_\lambda(k)(E_0+i\epsilon-\hbar^2k^2/2\mu)}, \tag{9}$$

where μ is the reduced mass of the colliding pair, and $\eta_\lambda(k)$ is the elastic scattering amplitude for the partial wave λ at the momentum k. This Green's function is based on the orthogonality relation for the optical-model radial wave functions,[15]

$$\int_0^\infty f_\lambda(k,r)f_\lambda(k',r)dr=\tfrac{1}{2}\pi\eta_\lambda(k)\delta(k-k'). \tag{10}$$

Actually, the optical-model regular radial wave functions $f_\lambda(k,r)$ do not, in themselves, form a complete set for all functions of r, and it is for this reason that Eq. (9) is written as an approximate equality. In principle, second-order transitions could use as intermediate states those channels to which the optical model "absorption" is taking place. Projection on to the low, vibrational states of the target nucleus does not eliminate this effect. However, it is argued that the simple nature of the operator $Y_k{}^qU'$ does not give much coupling of the entrance channel to the more complicated sorts of radial functions, so that for the present application the absorptive channels need not be carried in the Green's function. This approximation implies that the transitions induced by $Y_k{}^qU'$ are not the most important contributors to the imaginary part of the optical potential. It implies that the optical potential is the same in all the levels being treated here. Some further discussion of Eq. (9) is given in the Appendix.

With the use of Eq. (9), $\mathcal{T}_2$ takes the form

$$\mathcal{T}_2\approx -\frac{2}{\pi}\int_0^\infty \frac{\mathcal{S}(k,\Theta)}{E_0+i\epsilon-\hbar^2k^2/2\mu}dk, \tag{11}$$

and may be expressed in terms of a delta function term plus a principal value integral,

$$\mathcal{T}_2\approx -i\left(\frac{2\mu}{\hbar^2k_0}\right)\mathcal{S}(k_0,\Theta)-\left(\frac{2\mu}{\hbar^2k_0}\right)\frac{1}{\pi}P\int_0^\infty\frac{\mathcal{S}(k,\Theta)}{k-k_0}dk. \tag{12}$$

[15] In Eq. (10) an error in footnote 15 of reference 2 is corrected.

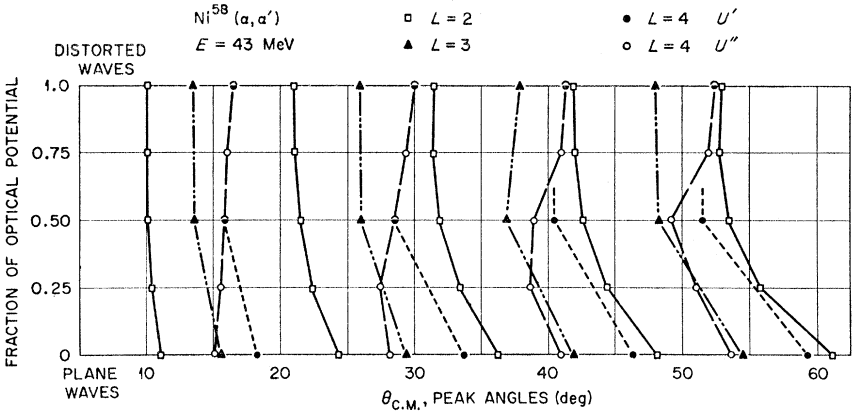

FIG. 2. Angles at which peaks occur in the angular distributions of alpha particles from Ni^{58} for various angular momentum transfers L. Calculations were made with optical potentials and Coulomb fields of 3/4, 1/2, and 1/4 of the "correct" strength, as well as with full strength, and with plane waves. All cases show the normal phase relation except the $L=4$ using U'' as interaction, which changes phase on going from plane waves to distorted waves.

In the principal value integral it has been assumed that $k\approx k_0$ will matter most, so $(k+k_0)\approx 2k_0$ has been introduced as a simplification. The two terms of Eq. (12) play very different roles in the determination of the cross section.

Explicit evaluation of $\mathcal{S}(k,\Theta)$ yields

$$\mathcal{S}(k,\Theta)=(4\pi/k_0k)\times\sum_{ll'} i^{l-l'}(2l+1)[(2I+1)/(2l'+1)]^{\frac{1}{2}}\times e^{i(\sigma_l+\sigma_{l'})}Y_{l'}{}^{M}(\Theta,0)\langle lI0M\,|\,l'M\rangle\langle lI00\,|\,l'0\rangle\times\{(V_{l'l'}V_{ll'}/\eta_{l'})+(V_{l'l}V_{ll}/\eta_l)\},\quad(13)$$

where

$$V_{ll'}(k_0,k)\equiv\int_0^\infty f_l(k_0,r)f_{l'}(k,r)U'dr.\quad(14)$$

It is at this stage that the consequences of the approximate grouping of spherical harmonics, used to go from Eq. (4) to Eqs. (5), become clear. The approximation has made it possible to perform the sum over λ, ν which appears in the Green's function, and these quantum numbers no longer appear. In the exact expression spherical harmonics belonging to different λ would have different radial factors. Our approximation replaces these different radial factors by one average value, permitting the sum over λ. If $k=2$, then if $l=l'$ the allowed values of λ would be l, $l\pm2$, whereas the approximation selects the one radial factor for $\lambda=l$. If $l=l'+2$ then $\lambda=l$, $l-2$, whereas the approximation uses an average of the radial factors for these two λ values. If $l=l'+4$ then $\lambda=l-2$, uniquely, but the approximation uses an average of the radial factors for l, $l-4$. This discussion would seem to make the approximation look good.

The factors η_l and $\eta_{l'}$ in the denominators in Eq. (13) become very small for l, l' much less than the cutoff angular momentum, $l_0(k)=kR_0$. However, the numerators are quadratic in the radial integrals $V_{ll'}$, which also become small at small l, l'. The numerical magnitude of the radial factors may be estimated fairly accurately by using an equation given by Austern and Blair[9] for $k=k_0$, the case of principal interest. It is

$$V_{ll}(k,k)=-(iE/2k)(\partial\eta_l/\partial R_0),\quad(15)$$

where R_0 is the nuclear radius. This equation merely is the expression for the rate of change of the scattering amplitude as the potential is changed. It immediately shows that the low l, l' terms of Eq. (13) are unimportant, despite the η_l denominators, so that localization of the important terms of the series near l_0 is preserved. In addition, a simplification of Eq. (13) is obtained. The sharp-cutoff property, well known for the function η_l, implies

$$\partial\eta_l/\partial R_0\approx-k\partial\eta_l/\partial l,\quad(16)$$

and therefore

$$(V_{ll}/\eta_l)\approx-\tfrac{1}{2}iE\eta_l{}^{-1}(\partial\eta_l/\partial l).\quad(17)$$

For l values near but below the sharp cutoff l_0, the logarithmic derivative of η_l is approximately constant, say 2ξ, where $2\xi\approx1$. For $l>l_0$ the logarithmic derivative goes to zero. When such a function multiplies $V_{ll'}$, the other radial-integral factor in Eq. (13), it displaces the rather localized peak in $V_{ll'}$ toward smaller l values, but does not change the basic structure of this peak. The displacement is of the order of half the peak breadth, and may be regarded as negligible. The displaced peak has a height of the order of ξ times its original height. With these approximations the logarithmic derivative may be factored out from $\mathcal{S}$, leaving

$$\mathcal{S}(k,\Theta)\approx(\tfrac{1}{2}iE\xi)(4\pi/kk_0)\times\sum_{ll'} i^{l-l'}(2l+1)[(2I+1)/(2l'+1)]^{\frac{1}{2}}e^{i(\sigma_l+\sigma_{l'})}\times Y_{l'}{}^{M}(\Theta,0)\langle lI0M\,|\,l'M\rangle\langle lI00\,|\,l'0\rangle\times[V_{ll'}+V_{l'l}].\quad(18)$$

This expression has the same basic structure in l, l' as does the term $\mathcal{T}_1$, discussed previously; hence it obeys the same phase rule as $\mathcal{T}_1$, the usual phase rule for angular momentum I.

A further simplification of $\mathcal{S}$ appears if $k=k_0$, in which case the two radial integrals become equal, and

$\mathcal{S}$ becomes a simple multiple of $\mathcal{T}_1$. A new expression for $\mathcal{T}_1$, therefore, may be given in terms of $\mathcal{S}(k_0,\Theta)$, by employing Eqs. (18), and (8) in Eq. (7a),

$$\mathcal{T}_1 \approx -[\mathcal{S}(k_0,\Theta)/iE\xi a]. \tag{19}$$

The entire transition amplitude then is expressed in terms of the function $\mathcal{S}$,

$$\mathcal{T}_1{}^M+\mathcal{T}_2{}^M \approx \left[-\left(\frac{2\mu i}{\hbar^2 k_0}\right)+\left(\frac{i}{E\xi a}\right)\right]\mathcal{S}^M(k_0,\Theta) - \left(\frac{2\mu}{\pi\hbar^2 k_0}\right)P\int_0^\infty \frac{\mathcal{S}^M(k,\Theta)}{k-k_0}dk. \tag{20}$$

It is easy to study the properties of this approximate equation.

Equation (20) shows how agreement with experiment must be achieved. Because $\mathcal{S}$ itself does not have the anomalous phase which is characteristic of double excitation, it is only possible to get agreement with experiment if the first term of Eq. (20) should be small, while the second have the anomalous phase. Indeed, very little need be known about $\mathcal{S}$ in order to see that the principal value integral does reverse the phase, giving the essential ingredient for agreement with experiment. It is more difficult to see under what conditions the first term of Eq. (20) will be small enough so that the term with anomalous phase may show itself.[16]

The ratio of magnitudes of the two parts of the first term of Eq. (20) is

$$(1/E\xi a) \div (2\mu/\hbar^2 k_0) = (1/k_0 a\xi). \tag{21}$$

For 40-MeV alpha particles $k_0 a \sim 2$, and $\xi \sim 1/2$, so the ratio is approximately equal to one, and the two parts of this first term do approximately cancel each other. This cancellation is independent of energy if $k_0\xi$ is independent of energy, a property which is not in disagreement with optical-model calculations. Of course, the present estimate only makes the cancellation plausible, because crude approximations have been used. However, to get agreement with experiment the first term of Eq. (20) must very nearly vanish, and the possibility is already seen to be sufficiently close that that result is believable.

The second term of Eq. (20) may be evaluated on the basis that $\mathcal{S}(k,\Theta)$ is an oscillating function of Θ, of which the oscillations are known to shift their phase as a function of k. Little more need be said. However, it is useful to introduce a functional description of $\mathcal{S}$ which makes the evaluation of the principal value integral more obvious. If Θ is not too close to 0° a good approximation is

$$\mathcal{S}(k,\Theta) \approx \mathcal{S}_0(k_0,\Theta)e^{-\gamma R_0|k-k_0|} \times \sin[\alpha(k-k_0)R_0\Theta + k_0R_0\Theta + \delta]. \tag{22}$$

The function $\mathcal{S}_0$ describes the envelope of the oscillations, and simply falls with Θ in the well-known way. The exponential factor describes the fact that $\mathcal{S}$ must also fall monotonically with $|k-k_0|$. Any other function with this same property gives the same general result. The quantity γR_0 is approximately the extent of the region of importance in the radial integral $V_{l_0 l_0}$, therefore, $\gamma \approx \frac{1}{2}$ very roughly. A plausible value for the parameter α is $\alpha \approx \frac{1}{2}$. Upon employing Eq. (22) in Eq. (20) the part of $\mathcal{S}$ which is even in $(k-k_0)$ does not contribute to the principal value integral, and the result of integration then is

$$\frac{1}{\pi}P\int_0^\infty \frac{\mathcal{S}(k,\Theta)}{k-k_0}dk \approx -\frac{2}{\pi}\mathcal{S}_0 \tan^{-1}(\alpha\Theta/\gamma)\cos(k_0R_0\Theta+\delta) \approx -\frac{2}{\pi}\tan^{-1}(\alpha\Theta/\gamma)\mathcal{S}'(k_0,\Theta). \tag{23}$$

Here, the function $\mathcal{S}'$ is defined as being obtained from $\mathcal{S}$ by a 90° phase shift of the oscillations. The origin of the anomalous phase in double excitation now is evident. The principle value integral selects a function with shifted phase.

Our final expression for T_{fi} now is

$$T_{fi}{}^M \approx -\beta^2 R_0{}^2 C(2\mu/\pi\hbar^2 k_0)(\tan^{-1}\Theta)\mathcal{S}'(k_0,\Theta), \tag{24}$$

where the rough approximation $(\alpha/\gamma)\approx 1$ has been introduced, and the first term of Eq. (20) has been dropped altogether. Over the range of values of the scattering angle Θ which are of experimental interest the function $\tan^{-1}\Theta$ continues to increase with Θ. In particular, this function does not get too close to its asymptotic value of $(\pi/2)$. Thus, the double-excitation cross section drops off less rapidly with angle than does the function $\mathcal{S}$, in agreement with experiment. Gross changes of parameter values would be required to alter this result.

The amplitude for single-phonon excitation is approximately

$$\beta R_0 C'(2/iE\xi)\mathcal{S}(k_0,\Theta), \tag{25}$$

where C' is a combination of Clebsch-Gordan factors, of the same nature and order of magnitude as C in Eq. (24). Upon combining Eqs. (24) and (25) the ratio of the magnitude of the two-phonon cross section to that of the one-phonon cross section is found to be, roughly,

$$(\tfrac{1}{2}\beta R_0 k_0 \xi \tan^{-1}\Theta)^2. \tag{26}$$

Typical parameter values which may be used in Eq. (26) are $\beta \approx 0.2$, $k_0R_0 \approx 17$, $\xi \approx 1/2$. Then the numerical value of Eq. (26) becomes

$$0.8\ (\tan^{-1}\Theta)^2. \tag{27}$$

[16] It is easy to show that the same ratios between the parts of (20) appear if we consider instead the excitation of a 4^+ *rotational* state of a nucleus with quadrupole deformation β, although the over-all cross section is reduced by (5/7).

A more correct calculation would give a slightly smaller value, because $(C'/C)<1$. Evidently at large Θ the double-excitation cross section is still about an order of magnitude smaller than that for one-phonon excitation. This also agrees with experiment.

III. CONCLUSIONS

Our calculations make it quite convincing that the transition amplitude has a part with the anomalous phase, that this part has an envelope which falls slowly with angle, and that it has approximately the right magnitude to agree with experiment. Whether the remainder of the transition amplitude is small enough so this part can show itself depends on a cancellation of the amplitude for simultaneous excitation of two phonons by the energy-conserving part of the amplitude for successive excitation. Of course, the cancellation need not be complete, because the experiments on double-excitation do not show very deep minima in the oscillations of the angular distribution.

Here we have only shown the cancellation to be plausible numerically, but its existence is required by experiment and by comparison with the adiabatic results.[7–9] More accurate calculations by Buck have been started, which perform numerical integrations and which treat the 0^+, 2^+, 4^+ wave functions in a coupled fashion, and preliminary results have been published.[12] These establish that cancellation does occur in the way we predict. Such calculations will be able to explore the sensitivity of the cancellation effect to the details of the physical model used, and to the details of the optical potential; and it may be that new physical information will thereby be obtained.

The adiabatic theory implicitly assumes a physical model having the cancellation property we have studied. In the rather accurate version of this theory given by Austern and Blair[9] it is made very plausible that the theory is adequate for the spectroscopic analysis of experiment. Nevertheless, the present article shows that other methods of analysis can obtain information which the adiabatic theory does not envisage.

Why the cancellation occurs may be related to the very strong interaction which alpha particles have with the deformed part of the optical potential. It has been suggested that the role played here by the energy-conserving term of the amplitude for successive excitation represents a depletion of the relevant part of the incident wave function by excitation of the single-phonon 2^+ state.[17]

ACKNOWLEDGMENTS

We are grateful to J. S. Blair, R. Lemmer, and N. S. Wall for communicating their work in advance of publication, and for initiating our interest in this problem. We also are grateful to J. S. Blair for permission to use his more recent results in advance of publication, and to him and N. S. Wall and J. E. Young for many other useful discussions, conducted in Copenhagen, and to R. H. Bassel for discussion and assistance with the machine calculations in Oak Ridge. One of us (N. A.) wishes to thank the Institute for Theoretical Physics of the University of Copenhagen for the hospitality and interest which has permitted much of this work.

APPENDIX

Outside the region of strong nuclear interactions the missing terms of the approximate Green's function, Eq. (9), no longer contribute, and that equation becomes exact. It is interesting to evaluate Eq. (9) in the limit $r\sim\infty$, to learn the structure of the asymptotic Green's function.

As r becomes very large, the function $f_\lambda(k,r)$ becomes a very rapid function of k. The function $f_\lambda(k,r')$ remains a slow function of k, because r' is limited to the region of overlap with internal nuclear wave functions, and therefore must remain small. Accordingly, the integral over k particularly concerns the function $f_\lambda(k,r)$. If only the rapidly varying factors of the integral over k are considered, then we must discuss

$$\lim_{r\to\infty}\int_0^\infty \frac{f_\lambda(k,r)dk}{E+i\epsilon-\hbar^2k^2/2\mu}=\lim_{r\to\infty}\left(\frac{2\mu}{\hbar^2}\right)\int_0^\infty \frac{f_\lambda(k,r)dk}{k_0^2+i\epsilon-k^2}.$$

Asymptotically, the function f_λ becomes[2]

$$f_\lambda\sim\tfrac{1}{2}i(e^{-i\theta_\lambda}-\eta_\lambda e^{i\theta_\lambda}),$$

where

$$\theta_\lambda\equiv kr-n\ln(2kr)-\tfrac{1}{2}\lambda\pi+\sigma_\lambda.$$

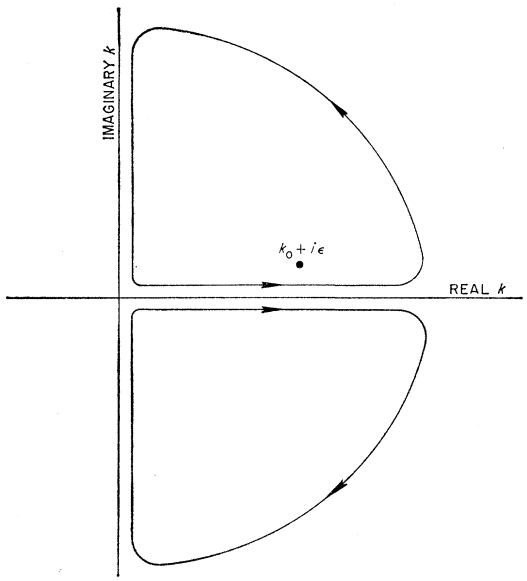

FIG. 3. Path of integration used in evaluating the integral (28).

[17] G. E. Brown (private communication).

The integral under consideration, therefore, is

$$\lim_{r\to\infty}\left(\frac{i\mu}{\hbar^2}\right)\int_0^\infty \frac{e^{-i\theta_\lambda}-\eta_\lambda e^{i\theta_\lambda}}{k_0^2+i\epsilon-k^2}dk. \tag{28}$$

The two terms of this integral may be studied in the complex plane, by suitably closing and deforming the path of integration, initially along the positive real axis. Figure 3 shows how the closed paths must be chosen, encircling the first and fourth quadrants. For sufficiently large r the portions of path along the imaginary k axis and along the circle at infinity do not contribute, provided the lower path is used for the term $\exp(-i\theta_\lambda)$ and the upper path for $\exp(i\theta_\lambda)$. Then only the second term leads to a nonvanishing residue from the one pole at $k=k_0+i\epsilon$. We find

$$\lim_{r\to\infty}\left(\frac{2\mu}{\hbar^2}\right)\int_0^\infty \frac{f_\lambda(k,r)dk}{k_0^2+i\epsilon-k^2}=\left(\frac{2\mu}{\hbar^2}\right)\tfrac{1}{2}i\left(-\frac{2\pi i}{2k_0}\right)(-\eta_\lambda e^{i\theta_\lambda(k_0)}).$$

The asymptotic Green's function, therefore, is

$$(E+i\epsilon-H_0)^{-1}\sim-(2\mu/\hbar^2)\exp[k_0r-n\ln(2k_0r)]/k_0rr'\times\sum_{\lambda\nu}e^{i(\sigma_\lambda-\frac{1}{2}\lambda\pi)}f_\lambda(k_0,r')Y_\lambda{}^\nu(\Omega)Y_\lambda{}^{\nu*}(\Omega').$$

In the second line of this expression we recognize the familiar "time-reversed final-state wave function," denoted in the present article by $\chi_f{}^{(-)*}$. The above discussion is interesting in that it shows the origins of the various factors of the familiar expression and, in particular, the manner in which the Coulomb phase factor enters this expression.

PHYSICAL REVIEW VOLUME 128, NUMBER 2 OCTOBER 15, 1962

Effect of the Optical Potential on the Nucleon Momentum Distribution in Nuclei. The Pickup Reaction*

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(Received June 6, 1962)

The effect of the optical potential on the nucleon momentum distribution in nuclei is discussed in this work. The idea of McCarthy *et al.*, that the refraction and localization effects caused by the real and imaginary parts of the optical potential smears the single-particle momentum distribution, is amplified and applied to nucleon momentum distribution experiments. The assertion is made here that it seems impossible to directly measure in any manner the momentum distribution of nucleons in nuclei. Further it is proposed that much or all of the large discrepancy between the experimental momentum determinations and the shell-model predictions is due to the neglect of these important sources of high-momentum components. The high-energy $C^{12}(p,d)C^{11}$ pickup reaction data are re-analyzed in the light of these considerations. It is shown that for $q^2/\beta^2\lesssim 8$, the $1p$-shell harmonic oscillator distribution, $(q^2/\beta^2)\exp(-q^2/\beta^2)$, has sufficient high-momentum components to fit the data. The lack of agreement for $q^2/\beta^2>8$ is possibly due more to a failure in the distorted-wave approximation calculation used here than to lack of high-momentum components in the wave function.

I. INTRODUCTION

THE success of the shell model for predictions of various properties of nuclei such as energy levels, magnetic moments, etc., has been amply verified by many workers. However, it has not been so successful in predicting the momentum distribution of ground-state nucleons in target nuclei as found by various nuclear reaction studies.[1–18] In general, the experimental results require much larger amounts, by orders of magnitude, of high-momentum components in the single-particle ground-state wave function than are supplied by the shell-model wave functions. This

* Based on work performed under the auspices of the U. S. Atomic Energy Commission.

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