# Pion Resonances*

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A self-consistent calculation of some features of the low-energy pion resonances is performed by a "bootstrap" method which preserves unitarity and the analytic properties of scattering amplitudes and satisfies crossing symmetry approximately. The interaction of two-pion states with pion-omega meson states is decisive for the properties of the $\rho$ meson. Values of the $\rho$-meson mass and width and the $\omega$-meson width are obtained in terms of the pion and $\omega$-meson masses and are in fair agreement with experiment.

## I. INTRODUCTION

As more and more of the new unstable particles or resonances are discovered, it is becoming increasingly clear that at least some of these particles, as well as some of those previously known, must be considered to be composite rather than elementary. Chew and Frautschi[1] extend this viewpoint to its logical limit and suggest that *all* the strongly interacting particles are composite. Every particle is assumed to be a stable or unstable combination of other particles. The forces which produce these compound states are themselves supposed to be caused by the exchange of particles. Thus, one abandons the notion that a dynamical theory logically begins with the specification of "elementary" particles and "fundamental" interactions. One has, rather, a situation in which the system of particles produces itself, in that the various particles give rise to forces among themselves making bound states which are the particles. We illustrate this concept by the following simplified example.[2] The $\rho$ meson appears as a resonance in the $\pi\pi$ system in the $J=1$, $T=1$ state. It is easy to see that the exchange of a $\rho$ meson between two pions yields an attractive force in the $J=1$, $T=1$ state. If the parameters of the $\rho$ meson are judiciously chosen, the attraction gives rise to a resonance whose mass and width are precisely those assigned to the $\rho$ meson. The $\rho$ meson has therefore produced itself, so to speak.

The actual physical situation is, of course, much more complicated. There are many strongly interacting particles, and they certainly all influence one another to a greater or lesser degree. For this reason, practical calculations of the properties of strongly interacting particles within the framework of this self-generating, self-consistent mechanism are very difficult. In order to reduce a problem to feasible proportions, many simplifications have to be made. One may wish to treat some parameters (e.g., the pion mass in the above example) as fixed by experiment and to apply the self-consistency requirement only toward the determination of the remaining parameters. Moreover, one may hope that in treating phenomena in a certain energy range, the role of particles with substantially higher rest energies is comparatively unimportant. Thus, when analyzing the contributions to the spectral integrals over energy for a partial-wave amplitude, one notices that a channel containing a "high"-mass particle has a threshold "far" to the right, while the force due to exchange of a "high"-mass particle contributes "far" to the left. The illustration involving the $\rho$ meson given above is an extreme example of this type of limitation.

In addition, it now seems probable that composite particles are described in terms of the so-called Regge philosophy, in which each composite particle is assigned an energy-dependent variable spin and coupling constant, which agree with the actual spin and usual coupling constant only when the particle is on the mass shell.[3] This means that each composite particle is characterized by energy-dependent functions instead of constants as for an "elementary" particle, and a self-consistent calculation of the properties of the particle should yield those functions. As this is harder than simply finding a mass and some coupling constants, another simplification would be to ignore, insofar as possible, the Regge behavior. One of the most striking features of the Regge behavior is that it seems to make scattering amplitudes less singular at large energies than would otherwise be the case. A natural cutoff is provided on quantities that might otherwise be divergent. One may hope that most of the Regge behavior can be summarized in a phenomenological cutoff, and that the rest does not make too much difference to semiquantitative self-consistent calculations.

Finally, even within the above simplifications, it is still necessary to make an approximation in calculating the position of bound states and resonances from a given

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[1] G. F. Chew and S. C. Frautschi, Phys. Rev. Letters **7**, 394 (1961).
[2] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961) and erratum p. 268. The definition of the coupling constant in this reference differs from the present one by a factor of 2.
[3] R. Blankenbecler and M. L. Goldberger, Phys. Rev. **126**, 766 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962); see also reference 1.

input force. Here the well-known $N/D$ method[4] seems to be the simplest to use, because it is a formalism that guarantees the correct unitarity and analyticity of partial-wave scattering amplitudes and yet is amenable to calculation.

A number of attempts[5] have been made to calculate the mass and width of the $\rho$ meson basically within the framework of the above approximations. Blankenbecler[6] has also tried to estimate very crudely the effect on the $\rho$ meson of the next most likely candidate as an influential particle, the $\omega$ meson with $J=1$, $T=0$ and negative $G$ parity. The main thing that has been learned from these calculations is that a self-generating $\rho$ meson is apparently possible; however, quantitative estimates of its properties are not very good.

In this article we attempt a more systematic inclusion of a larger class of particles into the self-consistent framework describing the $\rho$ meson, still using basically the approximations outlined above.

Other particles enter into the problem either through scattering channels that mix with the $\pi\pi$ channel or by directly affecting the force between the pions. The masses in which we are interested are those near the experimental[7] $\rho$ mass of about 750 MeV, so the most influential other channels are likely to be those with a threshold near this mass. The only two-particle states with threshold less than 1 BeV which can communicate with two pions are $\pi\omega$ and $K\bar{K}$ with thresholds at 920 and 990 MeV, respectively.[8] The process $\pi\pi\rightarrow\pi\omega$ can go with the exchange of a $\rho$ and $\pi\pi\rightarrow K\bar{K}$ can go with the exchange of $K^*$.[9] This second reactions is somewhat like the elastic process $\pi\pi\rightarrow\pi\pi$ with a $\rho$ exchange, which is assumed to be the basic force in the simplified self-consistent model, except that the $K^*$ is slightly heavier (884 MeV) than the $\rho$ and that the $K^*K\pi$ coupling constant is presumably much less than the $\pi\pi\rho$ coupling constant (because of the small width for $K^*\rightarrow K+\pi$ compared to that for $\rho\rightarrow\pi+\pi$). For this reason, as well as because the three-channel problem is very difficult mathematically, we shall ignore the $K\bar{K}$ channel.

Multiparticle channels (e.g., $4\pi$) with low thresholds also exist, but they are generally less important than two-particle ones because of the rapidly rising two-particle phase space. We shall therefore ignore them as well.

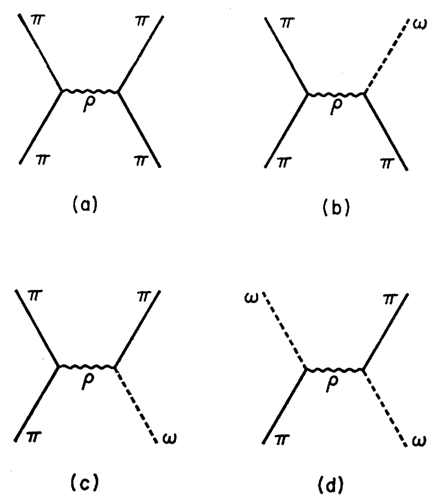


FIG. 1. Exchange graphs for the four processes $\pi\pi\rightarrow\pi\pi$, $\pi\pi\rightarrow\pi\omega$, $\pi\omega\rightarrow\pi\pi$, $\pi\omega\rightarrow\pi\omega$. These are the input.

We are thus led to a two-channel problem involving the four reactions $\pi\pi\rightarrow\pi\pi$, $\pi\pi\rightarrow\pi\omega$, $\pi\omega\rightarrow\pi\pi$, and $\pi\omega\rightarrow\pi\omega$, and we are primarily interested in the effect of the existence of the $\pi\omega$ channel on the self-consistency for the $\rho$ meson.

The forces that produce the various reactions are described in terms of exchange graphs in which a particle is exchanged among the scattering particles. There exist more complicated exchanges, involving several particles, which also contribute to the force. However, they have higher thresholds than one-particle exchanges, so we shall keep only these. It is easy to see that the only allowed one-particle exchanges in the four reactions are those shown in Fig. 1.[10] These graphs then constitute the input force.

We wish, in accord with the self-consistency idea, to require that this force give rise to the $\rho$ meson. This means the force should produce a resonance or bound state in the four processes, and, therefore, the four graphs of Fig. 2. These are the output.

The forces involve four parameters: the effective coupling constants $\gamma_{\rho\pi\pi}$ and $\gamma_{\rho\pi\omega}$ for the $\rho\pi\pi$ and $\rho\pi\omega$ vertices and the mass ratios $m_\rho/m_\pi$ and $m_\omega/m_\pi$. Given the force specified by these input parameters, we must calculate the energy and coupling constants of the resultant bound or resonant state, if it exists; that is, we calculate output values of $\gamma_{\rho\pi\pi}$, $\gamma_{\rho\pi\omega}$, and $m_\rho/m_\pi$. Hence, taking $m_\omega/m_\pi$ from experiment, we have a three-way self-consistency problem for the determination of the other three parameters.

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[4] M. Baker, Ann. Phys. (New York) 4, 271 (1958); G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[5] See reference 2, and Chew and Mandelstam (reference 4).

[6] R. Blankenbecler, Phys. Rev. **125**, 755 (1962).

[7] B. C. Maglić, L. W. Alvarez, A. H. Rosenfeld, and M. L. Stevenson, Phys. Rev. Letters **7**, 178 (1961).

[8] A meson of mass 550 MeV has also been observed by A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Straud, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, Phys. Rev. Letters **7**, 421 (1961). If this meson had the same quantum numbers as the $\omega$, it would, of course, also be expected to be quite important. There is some evidence, however [A. H. Rosenfeld, D. Carmony, and R. Van de Walle, *ibid.* **8**, 293 (1962)] that it is pseudoscalar with even $G$ parity, and if this is the case, it will not contribute strongly to the self-consistency of the $\rho$.

[9] M. H. Alston, L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho, and S. G. Wojcicki, Phys. Rev. Letters **6**, 300 (1961).

[10] On the basis of the Regge philosophy, it has been predicted that there may be a $J=2$, $T=0$, $\pi\pi$ resonance at around 1 BeV. If this actually turns out to exist, it may have a large effect on the $\rho$ meson. See C. Lovelace (to be published), and reference 3.

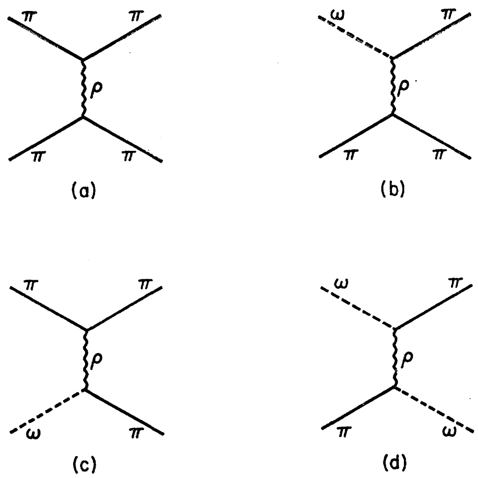


FIG. 2. The output graphs, describing the appearance of the $\rho$ as a resonance in the $\pi\pi$, $\pi\omega$ states.

In Sec. II, we shall describe in some detail how the $N/D$ method is to be applied to this problem, and what the self-consistency consists of. Section III contains a brief discussion of the calculations of the input forces with some limitations which these forces impose. The results are given in Sec. IV. Sections V and VI discuss the related problems of $s$-wave $\pi\pi$ scattering and the self-consistency for the $\omega$ meson, respectively. Finally, in Sec. VII, we summarize the conclusions which can be obtained.

## II. THE $N/D$ METHOD

For the sake of clarity, we shall begin with a brief review of the "$N/D$" method.[4] The purpose of this formalism is to have a framework within which unitarity and analyticity of the scattering amplitude are guaranteed whatever approximations are used in computing it. For a one-channel problem, the scattering amplitude $t(s)$ for a particular partial wave is a function of $s$, the total c.m. energy squared. It has a right-hand cut in the $s$ plane coming from unitarity and from "direct" graphs (for example, Fig. 2) and a left-hand cut due to "exchange graphs" (as in Fig. 1). If there are any stable one-particle states in this channel, $t(s)$ will have poles at the masses squared of such states. For $s$ above threshold, the unitarity condition for $t(s)$ states that

$$[t(s)-t^*(s)]/2i=t^*(s)t(s), \tag{2.1a}$$

or, equivalently,

$$\mathrm{Im}t^{-1}(s)=-1. \tag{2.1b}$$

Now these analyticity and unitarity properties are automatically satisfied if $t$ is represented as

$$t(s)=-N(s)/D(s), \tag{2.2}$$

where $D(s)$ has the same right-hand cut as $t$ but no left-hand cut, while $N(s)$ has the same left-hand cut as $t$ with no right-hand cut, and if

$$D(s)=1+\frac{1}{\pi}\int_{\text{threshold}}^{\infty}\frac{N(s')}{s'-s}ds'. \tag{2.3}$$

From (2.1), (2.2), and (2.3), it is clear that $N$ and $D$ may be determined from their respective dispersion relations and the following expressions for the absorptive parts:

$$\begin{aligned}\mathrm{Im}D(s)&=N(s),\\ \mathrm{Im}N(s)&=-D(s)\,\mathrm{Im}t_L(s),\end{aligned} \tag{2.4}$$

where $\mathrm{Im}t_L(s)$ is the discontinuity of $t(s)$ across its left-hand cut.

If one now uses an input $t_L(s)$, as determined, say, by a specified class of exchange diagrams, one has two coupled integral equations to solve for $N$ and $D$. One may think of solving these two integral equations by iteration. In the lowest order, then, one would use $D=1$ obtaining $\mathrm{Im}N=-\mathrm{Im}t_L$ and hence $N=-t_L$. Then $D$ is obtained as an integral over the input $t_L$, so that finally we get

$$t(s)=t_L(s)\left(1-\frac{1}{\pi}\int\frac{t_L(s')ds'}{s'-s}\right)^{-1}. \tag{2.5}$$

One could continue the iteration of the coupled $N$ and $D$ equations, if one so desired. However, it is clear that the process rapidly becomes more difficult, and, in any case, it is by no means clear that an exact solution to the $N$ and $D$ equations is a better approximation to $t$ than just the first iteration when the input $t_L$ is just some approximation to the left-hand cut in $t$ anyway. An exact solution guarantees that the input $\mathrm{Im}t_L$ remains the discontinuity across the left-hand cut of $t$, and if one uses as one input just a single diagram, say, one knows that the resulting $\mathrm{Im}t_L$ must physically be damped out as one goes far out on the left-hand cut. Hence, in the interests of simplicity, one may as well use only the first iteration.[11]

For our case, the one-channel problem consists in forgetting the $\pi\omega$ channel and keeping only the $\pi\pi$ channel. The input is taken to be the exchange of a $\rho$ meson between the two pions, as in Fig. 1(a). One may then compute the scattering amplitude $t(s)$ in the $J=1$, $T=1$ channel as outlined above, in the simple approximation, and ask that the $\rho$ meson itself appear as a resonance with the same parameters as those used in the input diagram. It turns out that there is a self-consistent solution, and the resulting $\rho$ meson parameters are found to be $m_\rho=350$ MeV, $\gamma_{\rho\pi\pi}^2/4\pi=0.6$.

As we have explained in the introduction, we are

[11] There are cases where the kinematics are such that use of the simplified method will lead to absurdities. For example, if the cut due to the exchange graph lies in the region where one is looking for a bound state, the simplified version would give a complex coupling constant. The integral-equation method, on the other hand, as can be seen from Eq. (2.4), will not, since $\mathrm{Im}N=0$ whenever $D=0$.

interested in including in this simple self-consistent calculation the effects of the (probably) next most important channel, that containing $\pi$ and $\omega$. We must modify the $N/D$ formalism to take account of two channels.

For two channels, the scattering amplitude becomes a $2\times 2$ matrix $t_{ij}(s)$ describing the amplitudes for the processes $\pi\pi\to\pi\pi$, $\pi\pi\to\pi\omega$, $\pi\omega\to\pi\pi$, $\pi\omega\to\pi\omega$. The unitarity condition reads

$$\frac{1}{2i}[t_{ij}(s)-t_{ij}^*(s)]=\sum_k t_{ik}^*(s)\theta(s-s_k)t_{kj}(s), \quad (2.6\text{a})$$

where $s_k$ is the threshold for the state $k$ and $\theta(s-s_k)$ is one or zero, according to whether $s$ is larger or smaller than $s_k$. If (2.6a) is multiplied by the inverse of the matrix $t$ on the right and by $(t^*)^{-1}$ on the left, we have

$$\text{Im}[t^{-1}(s)]_{ij}=-\delta_{ij}\theta(s-s_j). \quad (2.6\text{b})$$

It is quite straightforward to show that the matrix form

$$t_{ij}(s)=-\textstyle\sum_k N_{ik}(s)D_{kj}^{-1}(s), \quad (2.7)$$

where

$$D_{kj}(s)=\delta_{kj}+\frac{1}{\pi}\int_{s_k}^{\infty}\frac{N_{kj}(s')}{s'-s}ds' \quad (2.8)$$

automatically satisfies (2.6b) for any $N$.[12]

Now, if the simple approximation were used here, one would write $N_{ik}(s)\approx-(t_L(s))_{ik}$, where $t_L$ is the input from the exchange diagrams of Fig. 1. This approximation, while it of course preserves the unitarity and analyticity of $t_{ij}$, violates time reversal invariance, in that the approximate $t_{ij}$ computed in this way is not symmetric.[13] We may remark that even in a theory invariant under time reversal, $t_{ij}$ is symmetric only when the phases of the initial and final states are appropriately chosen. We suppose, throughout this paper, that this is done. One may try to guarantee the symmetry as well by using a form different from (2.7), but one that still is unitary. Such a form has been suggested by various authors, and is

$$t=-2(N^{-1}D+D^TN^{-1})^{-1}, \quad (2.9)$$

where $N$ and $D$ are still related by Eq. (2.8). The difficulty with this form is that if the coupling to the additional channel, containing $\pi$ and $\omega$ is made to vanish, the expression for $t$ does not reduce to the original expression for $t$ from the $\pi\pi$ channel alone.[14] As we shall see later, our final results are that the coupling constant linking the two channels is small, so it is certainly desirable to have this limit come out correctly.

Another form, which is also symmetric and unitary, and which does not suffer from this difficulty, is

$$t=-N^{1/2}D^{-1}N^{1/2}. \quad (2.10)$$

However, here one is required to deal with square roots of matrices, and the square roots apparently introduce undesirable singularities into the approximate $t$ matrix.

In the absence of any simple form which builds the symmetry into the approximation, and which does not suffer from other ugly diseases, we have elected to use the simple unsymmetrized form (2.7), in the hope that the final result will be nearly symmetric. Its departure from symmetry represents a measure of the degree to which the predictions of the approximate $N/D$ calculation differ from those of the exact calculation.

There are two more general points which must be discussed before we go into the details of our specific problem. The first is that the unitarity condition (2.6) actually extends below physical threshold for some of the reactions. Specifically, the reactions $\pi\pi\to\pi\omega$, $\pi\omega\to\pi\pi$, and $\pi\omega\to\pi\omega$ all have $s=(m_\pi+m_\omega)^2$ as physical threshold. Yet, in Eq. (2.6), their imaginary parts all get contributions from the $\pi\pi$ intermediate state, with a threshold at $s=4m_\pi^2$, and therefore exist down to this point. This results in the appearance of some kinematical singularities, which we remove as follows. Define $\bar{t}_{ij}=(p_ip_j)^{1/2}t_{ij}$, where $p_i$ and $p_j$ are the momenta of channels $i$ and $j$. Then, from Eq. (2.6b),

$$\text{Im}[\bar{t}^{-1}(s)]_{ij}=-\delta_{ij}\theta(s-s_j)/p_i. \quad (2.11)$$

We may now apply the $N/D$ method to $\bar{t}$ instead of to $t$; then

$$\bar{t}=-ND^{-1},$$

and

$$D_{ij}(s)=\delta_{ij}+\frac{1}{\pi}\int_{s_k}^{\infty}\frac{ds'}{s'-s}N_{ij}(s')\frac{1}{p_i}.$$

If the input is $t_L$, then to lowest order we have $N_{ij}=-(p_ip_j)^{1/2}(t_L)_{ij}$, and

$$D_{ij}(s)=\delta_{ij}-\frac{1}{\pi}\int_{s_i}^{\infty}\left(\frac{p_j}{p_i}\right)^{1/2}\frac{[t_L(s')]_{ij}}{s'-s}ds'. \quad (2.12)$$

Finally, $t=t_LD^{-1}$.

Secondly, we perform a subtraction in $D$. This amounts to factoring a constant out of $D$ by normalizing $D$ to unity at some point, and absorbing the constant into $N$, thus defining a new $N$.

The choice of subtraction point is connected with the requirement of crossing symmetry. Only the amplitude for the single $\rho$ exchange contributes to the discontinuity across the left-hand cut in the interval of that cut to the right of the thresholds for the exchange of higher mass systems. If the $D$ matrix is unity in this interval, then crossing symmetry tells us that the coupling constants used to calculate the forces due to the $\rho$ exchange are in fact the same as the coupling constants which determine the width and residues of the

[12] J. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960).

[13] Again, if the integral equation is solved exactly, this difficulty does not arise. See J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961).

[14] The form $N^{-1}$ becomes singular as the coupling to the second channel vanishes, and this leaves a residue in the expression for $t$. The fact that the coupling constant in the 22 element is like the square of that in the 12 element is the source of the difficulty.

resonant state, i.e., crossing symmetry determines this aspect of the self-consistency problem.

But in our simple calculation, $D$ can be fixed equal to unity at only one point in the interval of interest, the subtraction point. One may hope that $D$ approximates unity in the remainder of the interval. If this is true, crossing symmetry will be satisfied in an approximate manner.

Our procedure is, first of all, to find self-consistent parameters with the subtraction point placed at the endpoint of the left-hand cut. Secondly, we vary the subtraction point to determine the sensitivity of the results to its choice. If a calculated parameter varies within certain limits as the subtraction point ranges from the end point of the cut to, say, the threshold for two-$\rho$-meson exchange, then we say the parameter is predicted only to within the limits of that variation.

Thus, we will replace Eq. (2.12) by[15]

$$D_{ij}(s)=\delta_{ij}-\frac{s-s_0}{\pi}\int_{s_i}^{\infty}\left(\frac{p_j}{p_i}\right)^{1/2}\frac{[t_L(s')]_{ij}}{s'-s}\frac{ds'}{s'-s_0}. \quad (2.13)$$

With the subtraction at the beginning of the left-hand cut obtained from Fig. 1(a), $s_0=-m_\rho^2+4m_\pi^2$.

As we shall see later, even with one subtraction the integral for $D_{22}$ does not converge. This reflects the unrenormalizability of theories with *elementary* vector mesons. However, as discussed in the introduction, if we had treated the $\rho$ and $\omega$ as Regge particles, there would be a natural cutoff. We shall therefore imitate the Regge behavior by the introduction of an arbitrary cutoff on the $D_{22}$. Again, we hope that the result will be insensitive to this arbitrariness, and we must test this hope by varying the cutoff.

Let us conclude this section by outlining the steps in the calculation. This discussion presumes that a resonance with a narrow width will indeed be found. We first must compute $t_L(s)$. This is done from the diagrams shown in Fig. 1(a) through 1(d), which contribute to $(t_L)_{11}$ through $(t_L)_{22}$. (We denote the $\pi\pi$ state by index 1 and $\pi\omega$ by index 2.) Then $(t_L)_{ij}$ will depend on $x=s/4m_\pi^2$, $\alpha=m_\rho^2/4m_\pi^2$, $\beta=m_\omega^2/4m_\pi^2$, $\gamma_{\rho\pi\pi}$, and $\gamma_{\rho\pi\omega}$.

Next we compute $D(x)$ from Eq. (2.13); then $t(x)$ from

$$\bar{t}_{ij}(x)=\bar{t}_L(x)_{ik}\bar{D}(s)_{kj}/\det D(x), \quad (2.14)$$

where $\bar{D}$ is the cofactor matrix. Decompose $\det D(x)$ into real and imaginary parts: $\det D(x)=\Delta_R(x)+i\Delta_I(x)$, and write $\Delta_R'(x)$ for $(d/dx)\Delta_R(x)$. For a resonance at $x=\alpha$, we require $\Delta_R(\alpha)=0$, and for $x$ near $\alpha$,

$$\det D(x)\approx(x-\alpha)\Delta_R'(\alpha)+i\Delta_I(\alpha). \quad (2.15)$$

Then for $x$ near $\alpha$, $\bar{t}(x)$ has the general form

$$\bar{t}(x)=\begin{pmatrix} r_{11} & r_{12}\\ r_{21} & r_{22}\end{pmatrix}\Big/[(x-\alpha)+i\Delta_I(\alpha)/\Delta_R'(\alpha)]. \quad (2.16)$$

Anticipating that the width $\Gamma$ (in energy units) is small compared to $m_\rho$, we find that

$$m_\rho\Gamma/4m_\pi^2=\Delta_I(\alpha)/\Delta_R'(\alpha). \quad (2.17)$$

Moreover, the $r_{ij}$ may be taken to be real.[16] In particular,

$$r_{11}=[N_{12}(\alpha)\,\mathrm{Re}D_{21}(\alpha)-N_{11}(\alpha)\,\mathrm{Re}D_{22}(\alpha)]/\Delta_R'(\alpha), \quad (2.18a)$$

$$r_{22}=[N_{21}(\alpha)\,\mathrm{Re}D_{12}(\alpha)-N_{22}(\alpha)\,\mathrm{Re}D_{11}(\alpha)]/\Delta_R'(\alpha). \quad (2.18b)$$

Experimentally, $m_\rho<m_\omega$. If the calculation works satisfactorily, we expect $\alpha$ to be below the threshold for the $\pi\omega$ state, which we call $x_2$. Then $D_{22}(\alpha)$ and $D_{21}(\alpha)$ are real, while $D_{11}(\alpha)$ and $D_{12}(\alpha)$ have imaginary parts. By (2.17) and (2.18) we obtain

$$m_\rho\Gamma/4m_\pi^2=-r_{11}/p_1. \quad (2.19)$$

If the $\rho$ mass is above the $\pi\omega$ threshold, i.e., $\alpha>x_2$, then

$$m_\rho\Gamma/4m_\pi^2=-r_{11}/p_1-r_{22}/p_2, \quad (2.20)$$

which expresses the fact that if both channels are open to decay of the $\rho$, the total width is the sum of the partial widths for each channel.

Evidently, if $\alpha<x_1$, the lowest threshold, then $\mathrm{Im}[\det D(\alpha)]=0$ so that $\Gamma=0$, as it should be. For self-consistency $t_{ij}$ must agree, near $x=\alpha$, with the matrix $T_{ij}$ which represents the diagrams of Figs. 2(a) through 2(d). We find [see Eqs. (3.1) and (3.2) of the next section] for the matrix $\bar{T}_{ij}=(p_ip_j)^{1/2}T_{ij}$,

$$\bar{T}_{ij}=\frac{\begin{bmatrix} -\dfrac{1}{3}\dfrac{(\gamma_{\rho\pi\pi})^2p_1^4}{4\pi\ \ \alpha^{1/2}} & \dfrac{\sqrt{2}}{6}\dfrac{\gamma_{\rho\pi\pi}\gamma_{\rho\pi\omega}}{4\pi}(p_1p_2)^2\\ \dfrac{\sqrt{2}}{6}\dfrac{\gamma_{\rho\pi\pi}\gamma_{\rho\pi\omega}}{4\pi}(p_1p_2)^2 & -\dfrac{1}{6}\dfrac{(\gamma_{\rho\pi\omega})^2}{4\pi}p_2^4\alpha^{1/2}\end{bmatrix}}{[x-\alpha+i(m_\rho\Gamma/4m_\pi^2)]}, \quad (2.21)$$

where the momenta, at $x=\alpha$, are given by

$$(p_1/m_\pi)^2=\alpha-1,\quad (p_2/m_\pi)^2=\alpha+(\beta-\tfrac{1}{4})^2/\alpha-2(\beta+\tfrac{1}{4}).$$

The equations $\mathrm{Re}\det D(\alpha)=0$,

$$r_{11}=\bar{T}_{11}[x-\alpha+i(m_\rho\Gamma/4m_\pi^2)],$$
$$r_{22}=\bar{T}_{22}[x-\alpha+i(m_\rho\Gamma/4m_\pi^2)]$$

provide three relations among the four parameters $\alpha$, $\beta$ and the coupling constants which, if $\beta$ is specified to have its experimental value of[17] about 8, may allow the determination of the other three. If the self-consistent

[15] Incidentally, it is perfectly consistent to have a different subtraction point in each of the four matrix elements.

[16] We neglect a term $i\det N/\Delta_R'$ which is small if the width is small.

[17] The $\omega$ mass is, of course, not known with complete accuracy. A further blurring is introduced in that we neglect the difference between the $\pi^+$ and $\pi^0$ mass; this makes determination of the $\rho$ mass uncertain by perhaps 4%, i.e., 30 MeV.

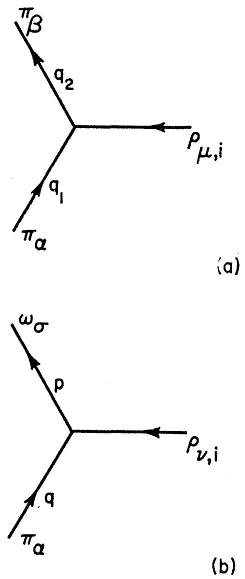


FIG. 3. The two basic vertex graphs for the $\pi\pi\rho$ and $\pi\rho\omega$ vertices.

mechanism is correct, a solution for $\alpha$ and the coupling constants should exist that gives more or less correct $\rho$-meson parameters.

It is clear that the residues should satisfy

$$\det r = r_{11}r_{22}-r_{12}r_{21}=0. \tag{2.22}$$

That they do so is easily verified. By definition of $\alpha$, Re $\det D(\alpha)=0$, and hence Re $\det\bar{D}(\alpha)=0$. Then, still assuming the width is small, $\det r$ is proportional to Re $\det\bar{D}(\alpha)$ and also vanishes.

The other condition on the residues, namely, that $r_{12}=r_{21}$, is not necessarily guaranteed by our formalism because we have not used one of the symmetric forms. The four residues, therefore, have only one relation between them in this approximation, instead of two. There are therefore three different quantities to be computed (in addition to $\alpha$) instead of two. We may take these to be $\gamma_{\rho\pi\pi}^2/4\pi$, $\gamma_{\rho\pi\omega}^2/4\pi$, and $r_{12}/(r_{11}r_{22})^{1/2}$, say. The first two are the desired coupling constants, and the last, by its deviation from unity, measures the amount of deviation from symmetry of the $t$ matrix.

## III. THE INPUT FORCES

In this section we shall write down the expressions for $t_L$ which result from the input diagrams of Figs. 1(a) to (d). The $\rho\pi\pi$ and $\rho\pi\omega$ vertices shown in Fig. 3 will be described in terms of the effective interactions

$$2i\gamma_{\rho\pi\pi}\epsilon_{\alpha\beta i}(q_1+q_2)_\mu, \tag{3.1}$$

and

$$(\gamma_{\rho\pi\omega}/m_\pi)\delta_{i\alpha}\epsilon_{\mu\lambda\sigma\nu}q_\mu p_\lambda, \tag{3.2}$$

where $\alpha$, $\beta$, $i$ are isotopic indices. These expressions serve to define $\gamma_{\rho\pi\pi}$ and $\gamma_{\rho\pi\omega}$. We are interested in the expression for the $J=1$, $T=1$ projection of the diagrams of Fig. 1(a) to (d), using these interactions. The calculation of these projections is straightforward though unpleasant, and we find the following results:

$$t_L(x)_{11}=\left(\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\right)\left(\frac{x-1}{x}\right)^{1/2}(2x+\alpha-1)\times\left[-\frac{2}{x-1}+\frac{x+2\alpha-1}{(x-1)^2}\ln\frac{x+\alpha-1}{\alpha}\right]; \tag{3.3}$$

$$t_L(x)_{12}=t_L(x)_{21}=\frac{1}{\sqrt{8}}\frac{\gamma_{\rho\pi\omega}\gamma_{\rho\pi\pi}}{4\pi}(qq')^{1/2}\times\left[2C+(1-C^2)\ln\frac{C+1}{C-1}\right], \tag{3.4}$$

where

$$q=(x-1)^{1/2},\quad q'=\left[x+\frac{(\beta-\frac{1}{4})^2}{4}-2(\beta+\tfrac{1}{4})\right]^{1/2},$$

and

$$C=-(x+2\alpha-\beta-\tfrac{3}{4})/(qq').$$

Next

$$t_L(x)_{22}=\frac{1}{16}\frac{\gamma_{\rho\pi\omega}^2}{4\pi}\frac{q'}{x^{1/2}}\left(\tfrac{1}{2}b_0\ln\frac{D+1}{D-1}+\bar{b}_0-\tfrac{1}{3}q'^2\right), \tag{3.5}$$

where

$$b_0=a_0-D\bar{b}_0,$$

$$\bar{b}_0=a_1-Da_2+D^2a_3,$$

and

$$a_0=q'^2+1,$$
$$a_1=q'^2+4x-4(\alpha-\tfrac{1}{4})^2/x,$$
$$a_2=3q'^2+8\alpha+1,$$
$$a_3=-q'^2,$$

and finally

$$D=\frac{x+2\alpha-2\beta-\frac{1}{2}-(\beta-\frac{1}{4})^2/x}{q'^2}.$$

The threshold for the $\pi\pi$ channel is $x_1=1$. The threshold for the $\pi\omega$ channel is $x_2=(\beta^{1/2}+\frac{1}{2})^2$.

Experimentally, the $\rho$ mass is so high that the decay $\omega\to\rho+\pi$ cannot occur. In our self-consistent calculation, however, we must consider the possibility of other $\rho$ masses. If a trial $\rho$ mass is sufficiently low, we can have $\omega\to\rho+\pi$. The possibility of this decay corresponds to the appearance of a singularity in the Feynman propagator for the exchanged $\rho$ of Fig. 1(b, c, or d) in the physical region. This singularity can only be removed by a proper treatment of the instability of the $\omega$ meson for decay into $\rho$ and $\omega$, and requires the introduction of the decay channel $\pi+\omega\to\pi+\pi+\rho$. The same difficulty crops up in the analysis of the $\pi\rho$ scattering which will be discussed in Sec. VI; we shall therefore content ourselves here with observing that the introduction of this additional channel is beyond the scope of our calculation. We must therefore restrict ourselves to regions of $\rho$ mass above the threshold for this decay.

This threshold is

$$(\beta^{1/2}-\tfrac{1}{2})^2,$$

so we must assume $\alpha>(\beta^{1/2}-\frac{1}{2})^2$. Since experimentally $\beta\approx 7.95$, this says we can only work for values of $\alpha\gtrsim 5.4$. If $\alpha$ comes out anywhere near its experimental value of about 7.3, this limitation does not cause any difficulty.

These exchange graphs all produce "left-hand cuts" in the partial-wave amplitudes $t_L$. The cut coming from Fig. 1(a) is $-\infty<x<1-\alpha$. From Fig. 1(d) there is a short segment for $(\beta-\frac{1}{4})^2/\alpha<x<2\beta+\frac{1}{2}-\alpha$. We are interested in evaluating $t_L$ at $x=\alpha$ to obtain the coupling constants: The cut thus exists if $(\beta-\frac{1}{4})^2<\alpha\ ^2$and if $2\alpha<2\beta+\frac{1}{2}$, i.e., if $\beta-\frac{1}{4}<\alpha<\beta+\frac{1}{4}$. In this region $t_L(\alpha)$ is complex and the residues are therefore complex. This means our approximation predicts complex coupling constants if $\alpha>\beta-\frac{1}{4}$. We must therefore restrict ourselves to $\alpha<\beta-\frac{1}{4}\approx 7.7$.

Altogether, then, we can see that the simple form of the $N/D$ approximation requires that $\alpha$ is in the range between 5.4 and 7.7. Otherwise, we get an inconsistency.[18]

The calculation from this point on requires the numerical integration of Eq. (2.13) using the $(t_L)_{ij}$ given in Eqs. (3.3) to (3.5). The computations were carried out on the IBM 704 at the University of California at Berkeley. The results are described in the next section.

### IV. RESULTS

In the uncoupled problem, where the $\pi\omega$ channel was neglected, the two conditions,

$$\mathrm{Re}D(\alpha)=0 \tag{4.1}$$

and

$$-\frac{1}{3}\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\frac{(\alpha-1)^{3/2}}{\sqrt{\alpha}}=-\frac{N(\alpha)}{(d/dx)\,\mathrm{Re}D(x)|_{x=\alpha}}, \tag{4.2}$$

gave two relations between $\gamma_{\rho\pi\pi}$ and $m_\rho/m_\pi$. These two relations are conveniently represented as two curves of $G=\gamma_{\rho\pi\pi}^2/4\pi$ vs $m_\rho$ in MeV, the intersection of which gives the self-consistent values. The curves are shown in Fig. 4.

The coupled problem, involving as it does self-consistency among three numbers, $G$, $m_\rho$, and $H\equiv\gamma_{\rho\pi\omega}^2/4\pi$, is somewhat more difficult to represent. To begin with, in analogy to the uncoupled case, one may plot for each fixed input $H$ the relation between the input $G$ and $m_\rho$ which makes Re det$D(\alpha)=0$. These are also shown in Fig. 4. They may be thought of as showing the self-consistent value of $m_\rho$ as a function of the input $G$ and $H$. We see that the effect of the existence of the $\pi\omega$ channel is attractive for masses greater than about 665 MeV, and repulsive for smaller masses. This is clear because for a given mass the $G$ and, hence, the attraction due to the $\rho$ exchange, necessary for self-consistency, grows with growing $H$ for smaller masses and decreases with growing $H$ for larger masses.

[18] If the integral equations for $N$ and $D$ were solved exactly, we could eliminate the upper restriction here, but not the lower one.

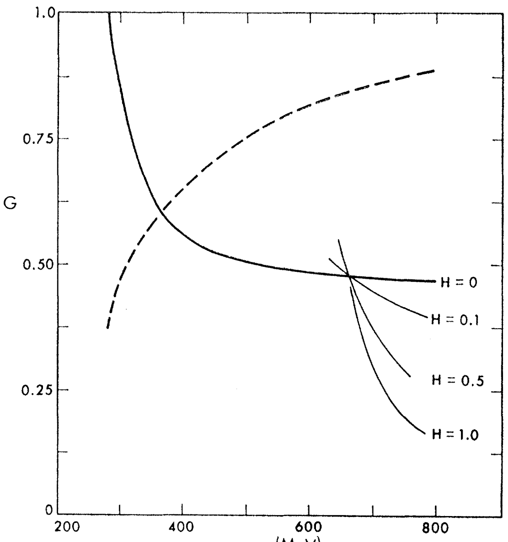


FIG. 4. The relation between $G=\gamma_{\rho\pi\pi}^2/4\pi$ and $m_\rho$ such that $m_\rho$ (input)$=m_\rho$ (output) is graphed for several $H$ values (solid lines). The dashed line represents the condition $G$ (input)$=G$ (output) for $H=0$. Self-consistency for the single-channel problem ($H=0$) is achieved for $G=0.6$, $m_\rho=350$ MeV.

To find the three self-consistent values of $G$, $H$, and $m_\rho$, we draw graphs as follows. For a given input mass $m_\rho$, there is one relation between the input $G$ and input $H$ in order that Re det$D=0$ at this mass. In addition, there is another relation between the input $G$ and input $H$ coming from the requirement that the output $G$, as computed from $r_{11}$ equal the input $G$. Finally, we get a third relation by requiring that the output $H$, as computed from $r_{22}$ agree with the input $H$. Then for each $m_\rho$, we have three $G$ vs $H$ curves which may be termed the det$=0$ curve, the $G=G$ curve, and the $H=H$ curve. The value of $m_\rho$ at which these three curves coincide at a point is the self-consistent $m_\rho$, and the coordinates of the intersection point in the $GH$ plane are the self-consistent $G$ and $H$.

Three of these $G$ vs $H$ graphs are shown in Figs. 5, 6, and 7, for $m_\rho=654$, 659, and 700 MeV.

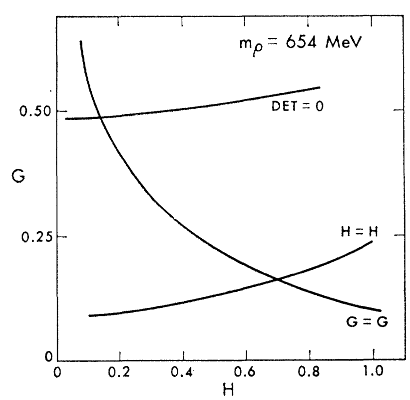


FIG. 5. The three consistency conditions for $m_\rho=654$ MeV. They cannot be simultaneously satisfied for this mass.

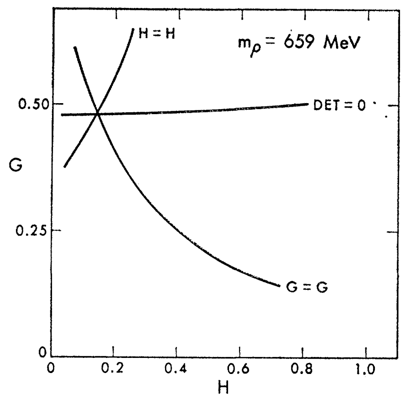


FIG. 6. The three consistency conditions for $m_\rho=659$ MeV. Consistency is achieved at $G=0.48$, $H=0.14$.

To the accuracy of the numerical calculation, self-consistency occurs at $m_\rho\approx 659$ MeV. The corresponding values of $G$ and $H$ are $G=0.48$, $H=0.14$.

The remaining number of interest has to do with the asymmetry. The self-consistent values are computed from the diagonal elements of $t_{ij}$ and from det$D$. If we had been able to use a symmetrized formalism, the off-diagonal elements would then be determined unambiguously. The lack of symmetry in our approximation may then be looked on as measured by the deviation of the off-diagonal residues from $(GH)^{1/2}$, which they should equal. (Of course, the fact that the product of the off-diagonal element equals $GH$ is guaranteed, as shown in Sec. II.) We find for the self-consistent parameters that the ratios of the two off-diagonal elements to $(GH)^{1/2}$ are 1.7 and 1/1.7 instead of 1 and 1. If we had been willing to use as an input not $(t_L)_{12}$ and $(t_L)_{21}$ but had altered these to $(t_L)_{12}\times 1.3$ and $(t_L)_{21}/1.3$, we would have obtained a self-consistent solution with the same parameters as above and with the same ratio for the off-diagonal elements.

These self-consistent numbers were computed with the subtraction point at $x=1-\alpha$, at the beginning of the left-hand cut produced by the exchange of a $\rho$ meson between two pions. The next cut is due to the exchange of $\pi$ and $\omega$, and it therefore begins at $1-(\beta+\frac{1}{2})^2\sim -10$. In principle, the argument that the subtraction point should be chosen to give approximate crossing symmetry merely says that the subtraction point should be anywhere between $1-\alpha$ and $-10$. We have therefore also carried out the self-consistent calculations with variable subtraction points. As the subtraction point is varied between $-3$ and $-10$, the derived mass changes by only a few percent, and $G$ varies between 0.55 for $-3$ and 0.31 for $-10$. $H$ varies between 0.12 and 0.19 over the same interval. The calculation is sufficiently insensitive to where the subtraction point is chosen.

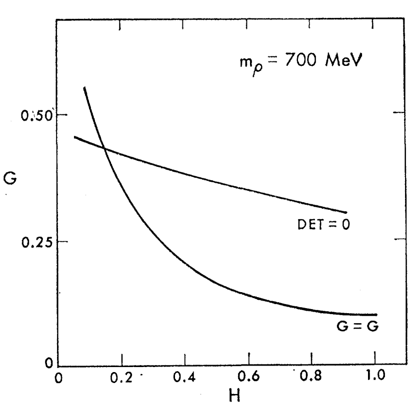


FIG. 7. The consistency conditions for $m_\rho=700$ MeV. There is no $H=H$ curve since the predictions for the output $H$ are all negative.

Because of the divergence of the $D_{22}$ integral, as remarked earlier, it was necessary to introduce a cutoff. The cutoff is presumably a crude representation of the natural cutoff provided by the Regge behavior of composite states. The numbers quoted above are for a cutoff at $x=35$, or about two nucleon masses. If this is varied to $x=200$, or around 5 nucleon masses, there is very little (a few percent) change in $m_\rho$, $G$ ,and $H$. The results are insensitive to the cutoff, so that its use seems justified.

One may ask how important are the contributions to the $D_{ij}$ integrals from high energies and how much channels with high-energy thresholds are likely to affect the $\rho$ parameters. To test this, we have repeated the calculation imposing arbitrarily a cutoff of about two nucleon masses on *all* the integrals. We then obtain $m_\rho=740$ MeV, $G=0.87$, $H=0.5$. Thus the high energies are important, and any physical effect that tends to reduce $N_{11}$, $N_{12}$, $N_{21}$ at high energies acts to raise $m_\rho$ toward the correct experimental value.

## V. S WAVES

The treatment of the $\rho$ as a particle whose exchange defines a force constitutes an approximation to the force defined by the exchange of a general 2-pion system in the $J=1$, $T=1$ state. Since the resonance is strong and well-defined, the approximation is expected to be a good one. One may inquire about the effect on our self-consistency calculation of strong pion-pion interaction in the $T=0$ and $T=2$ states. These are associated with even $J$ because the pions are bosons. The $\rho$ exchange produces an attractive force in the $T=0$ states, which is strongest for $J=0$, and produces a weaker, repulsive force in the $T=2$ states. Furthermore, experimental evidence for strong interactions in the $J=0$, $T=0$ state of two pions has been obtained by several groups.[19]

We assume that there is an actual resonance, say $\sigma$, with $J=0$, $T=0$ and even $G$ parity. Then this "particle" contributes to the interaction force between two pions in a $J=1$, $T=1$ state, as well as in the $J=0$, $T=0$ state.

[19] N. E. Booth, A. Abashian, and K. M. Crowe, Phys. Rev. Letters **7**, 35 (1961); B. C. Barish, R. J. Kurz, D. MacManigal, V. Perez-Mendez, and J. Solomon, *ibid.* **6**, 297 (1961); J. Kirz, J. Schwartz, and R. Tripp, Phys. Rev. **126**, 763 (1962).

Conversely, the $\rho$ contributes an attractive force to the $J=0$, $T=0$ state as well as to the $J=1$, $T=1$ state. This is illustrated in Fig. 8(a). There is thus a "double self-consistency" problem for these two "particles."

This problem is straightforward to set up, if we neglect the $\pi\omega$ channel. Our primary conclusion will be that the $S$-wave considerations do *not* affect the calculation of the $\rho$ parameters; then this qualitative result should not be affected by the presence or absence of the $\pi\omega$ channel. There are then two separate one-dimensional problems, one for each state. We can write

$$t_0=-N_0/D_0,\quad t_1=-N_1/D_1 \tag{5.1}$$

for each of the scattering amplitudes $t_0$ and $t_1$ for the $J=0$, $T=0$ and $J=1$, $T=1$ channels, with the usual relation between $N$ and $D$. As input we choose $N_0=-t_L^{00}$, $N_1=-t_L^{11}$, where $t_L^{00}$ and $t_L^{11}$ are derived from the diagrams of Fig. 8(a), projected into the 00 and 11 states, respectively.

Both $D_0$ and $D_1$, computed from this choice of $N$, will require subtractions, and we follow the philosophy described in the previous section.

In previous treatments of $\pi\pi$ interactions, a point interaction corresponding to Fig. 8(b) has been introduced, with an interaction strength denoted by $\lambda$. Presumably this $\lambda$ is really just a phenomenological way of describing the remaining forces not included in the simple exchange diagrams. Insofar as the self-consistency scheme is valid, the $\lambda$ should be relatively unimportant and could be ignored.

The input, from the diagrams of Fig. 8(a), is the following:

$$t_L^{00}=\left(\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\right)\left(\frac{x-1}{x}\right)^{1/2}\left(-1+\frac{2x+\alpha-1}{x-1}\ln\frac{x+\alpha-1}{\alpha}\right)$$
$$+\frac{1}{4}\left(\frac{f^2}{4\pi}\right)\left(\frac{x-1}{x}\right)^{1/2}\left(\frac{1}{x-1}\ln\frac{x+\delta-1}{\delta}\right); \tag{5.2}$$

$$t_L^{11}=\left(\frac{\gamma_{\rho\pi\pi}^2}{4\pi}\right)\left(\frac{x-1}{x}\right)^{1/2}(2x+\alpha-1)$$
$$\times\left[-\frac{2}{x-1}+\frac{x+2\alpha-1}{(x-1)^2}\ln\frac{x+\alpha-1}{\alpha}\right]$$
$$+\frac{1}{4}\left(\frac{f^2}{4\pi}\right)\left(\frac{x-1}{x}\right)^{1/2}$$
$$\times\left[-\frac{2}{x-1}+\frac{x+2\delta-1}{(x-1)^2}\ln\frac{x+\delta-1}{\delta}\right]. \tag{5.3}$$

Here $\delta=m_\sigma^2/4m_\pi^2$, and $f$ is an effective coupling constant for the $\sigma\pi\pi$ vertex.

The numerical solution of this coupled problem shows that the influence of the hypothetical $S$-wave resonance on the $P$ wave is very small and the resonance does not support itself in a self-consistent manner. We shall not consider the $S$-wave problem further here.

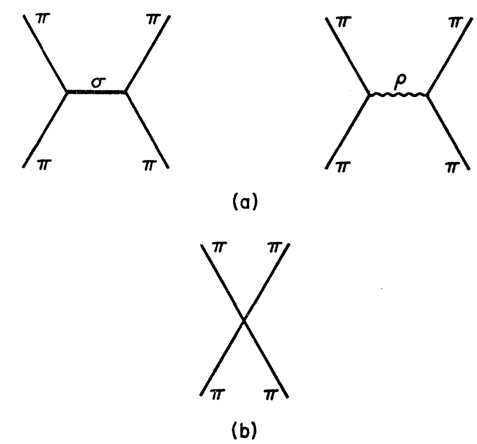


FIG. 8. Input graphs for the $S$-wave problem.

## VI. THE $\omega$ PROBLEM

If the $\rho$ is a self-generating particle, as has been assumed everywhere above, which is strongly influenced by the presence of the inelastic $\pi\omega$ channel with a threshold near the $\rho$ mass, then in reverse one might expect the $\omega$ to be a self-generating particle strongly influenced by the $\pi\rho$ channel with threshold near the $\omega$ mass.

The low-mass systems in which the $\omega$ will appear as a virtual state are $3\pi$ and $\pi\rho$. One might try, in accord with the idea of the importance of two-particle states, to keep just $\pi\rho$ as a beginning. The force between $\pi$ and $\rho$ then would come from the exchange of a $\pi$ or an $\omega$, as shown in Fig. 9(a). Here, however, we run into the difficulty first mentioned in Sec. III. Because the $\rho$ is unstable for the decay into $\pi+\pi$, this graph is singular in the physical region unless the unstable $\rho$ is described more correctly. We must put in the physically possible

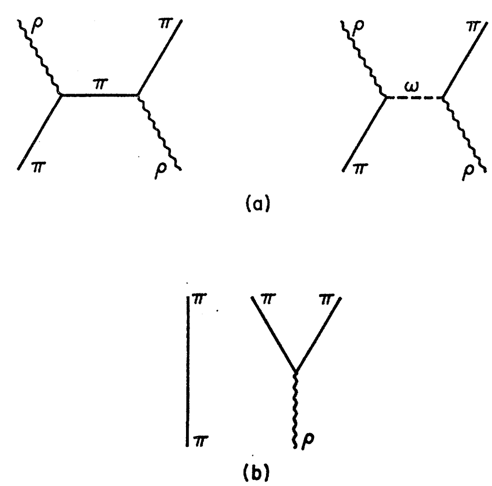


FIG. 9. Input graphs for the $\omega$ self-consistency problem.

decay, and hence must include the amplitude $\pi+\rho\rightarrow 3\pi$, as shown in Fig. 9(b).

The problem, therefore, is *necessarily* two channel in this formulation. It appears to be possible to take into account the $\rho$ decay consistently and then to reduce the problem to an equivalent one-channel formulation. Calculations along these lines are being carried out.

## VII. CONCLUSIONS

In conclusion, the following points may be emphasized. First of all, a self-consistent solution does exist. This result is not trivial. If our model were not a reasonable approximation to reality, there would be no reason to anticipate that self-consistency among three separate quantities would occur at all. No adjustable parameters were introduced into the theory to facilitate this self-consistency. Secondly, the effect of the $\omega$ channel is very large. The results for the $\rho$ mass and width are highly sensitive to $\gamma_{\rho\pi\omega}$. It seems clear, as Blankenbecler already has suggested, that any calculation of the $\rho$ meson that ignores the $\pi\omega$ channel cannot hope for success. Thirdly, since fairly high energies play a non-negligible role in the calculation, as noted in Sec. IV, the inclusion of additional channels may be needed for a truly close agreement with experiment.

The most recent experimental evidence[20] indicates $m_\rho=767$ MeV, and a full width $\Gamma=110$ MeV for the decay $\rho\rightarrow 2\pi$. Then

$$\gamma_{\rho\pi\pi}^2/4\pi=3[m_\rho^2/(m_\rho^2-4m_\pi^2)]^{3/2}(\Gamma/m_\rho)\approx 0.5.$$

About $\gamma_{\rho\pi\omega}^2/4\pi$, less is known; from the upper limit[21] of 24 MeV for the full width of $\omega\rightarrow 3\pi$, one can conclude[22] $\gamma_{\rho\pi\omega}^2/4\pi\lesssim 1$. All these numbers are in acceptable agreement with the self-consistent values of $m_\rho\sim 659$ MeV, $\gamma_{\rho\pi\pi}^2/4\pi\approx 0.3$ to 0.6, $\gamma_{\rho\pi\omega}^2/4\pi\approx 0.1$ to 0.2. Thus, our calculation yields the prediction that the $\omega$ width is several MeV.

In appraising the accuracy of our prediction of the $\rho$-meson mass, we must not overlook the approximate nature of the method, the remark in footnote 17, and the asymmetry of the $t$ matrix which presumably reflects in some way the errors of the calculation.

One of us (F. Z.) is indebted to the hospitality of the Lawrence Radiation Laboratory, Berkeley, where a major portion of this work was carried out.

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[21] L. M. Stevenson, L. W. Alvarez, B. C. Maglić, and A. H. Rosenfeld, Phys. Rev. **125**, 687 (1962).

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