Pion-Pion Scattering Including Inelastic Intermediate States

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The inverse amplitude dispersion relations for pion-pion scattering including inelastic intermediate states are solved for given models of the inelastic cross section. These models are calculated by representing the four-pion intermediate state as the combination of a three-pion resonance and a single pion. Interesting results are obtained using a small and slowly varying inelastic cross section, since this leads to a resonant behavior of $R=\sigma^{\text{tot}}/\sigma^{\text{el}}$ when the real part of the P-wave phase shift passes through π in the inelastic region. The original solution of the pure elastic inverse amplitude equations exhibits a single sharp resonance in the P wave, and the inclusion of a resonant R in the iteration scheme results in the appearance of a second P-wave resonance. When the pion-pion coupling constant $\lambda\cong-0.1$ and the three-pion decay coupling constant $G^2\cong1$, the positions and widths of the two di-pion resonances are $M_\zeta\cong600$ MeV, $\Gamma_\zeta\cong30$ MeV and $M_\rho\cong800$ MeV, $\Gamma_\rho\cong50$ MeV. This value of λ gives S waves in good agreement with independent analyses of π-N scattering data. These results prove that, at least for pion-pion scattering, solutions exist possessing two resonances with the same quantum numbers. When a rapidly rising inelastic cross section is used corresponding to a large G^2, the original elastic solution exhibiting a single P-wave resonance is changed inappreciably due to the dominance of the "nearby" inverse amplitude left cut.

THE inverse amplitude dispersion relations for pion-pion scattering[1–3] have been reformulated to include contributions from the inelastic intermediate states, and solved, as before, by numerical iteration. The four-pion intermediate state is approximated by a combination of a single-pion and a three-pion resonance, or of two di-pion resonances, and the resulting inelastic amplitude calculated by perturbation theory. With this model the four-pion intermediate state is pure isotopic spin $I=1$ at low energies, and thus only contributes directly to the P waves.

In the case of the two-pion exchange approximation, Bransden and Moffat[2,3] found P-wave dominant solutions possessing a single sharp resonance for the pion-pion coupling constant $-0.08\geq\lambda\geq-0.40$. When $|\lambda|\leq0.15$ the S-wave scattering lengths are consistent with pion-production data.[3,4] These P-wave dominant solutions satisfy the first and higher derivative crossing conditions. In none of the cases considered is this resonance in the P wave removed by the inclusion of inelastic effects, although in some cases one or more new P-wave resonances are produced. Thus, the original resonance in the P wave dominates the low- and intermediate-energy P-wave solutions. The S waves remain unaltered.

These results are only sensitive to the behavior of the inelastic amplitude near threshold. Among other results, we have obtained cases exhibiting two di-pion resonances of masses 600 MeV and 800 MeV, which could account for the existence of two particles with the same quantum numbers. Some evidence has been reported for the existence of the ζ particle with $I=1$ and possibly $J=1^-$ at an energy of 550–600 MeV.[5] The ρ particle with $I=1$, $J=1^-$ at 750–800 MeV is now well established, although its detailed structure is not yet known.[5]

Consider the P-wave amplitude $f(\nu)=A_1{}^1(\nu)$, which assumes the form in the physical region

$$f(\nu)=[(\nu+1)/\nu]^{\frac{1}{2}}(e^{2i\delta}-1)/2i. \tag{1}$$

Above the inelastic threshold ν_T, $\delta=\delta_R+i\delta_I$ with $\delta_I>0$ in virtue of unitarity. The variable $\nu=q^2/\mu_\pi{}^2$ with q^2 equal to the square of the barycentric momentum. We have

$$\operatorname{Im}f=[\nu/(\nu+1)]^{\frac{1}{2}}|f|^2+[(\nu+1)/\nu]^{\frac{1}{2}}(1-e^{-4\delta_I})/4, \tag{2}$$

and unitarity implies that in the physical region

$$\operatorname{Im}(f^{-1})=-R[\nu/(\nu+1)]^{\frac{1}{2}}, \tag{3}$$

with $R=\sigma^{\text{tot}}/\sigma^{\text{el}}$. The dispersion relation for $\operatorname{Re}(f^{-1})$

[1] J. W. Moffat, Phys. Rev. **121**, 926 (1961).
[2] B. H. Bransden and J. W. Moffat, Nuovo cimento **21**, 505 (1961).
[3] B. H. Bransden and J. W. Moffat, Phys. Rev. Letters **8**, 145 (1962).
[4] A recent analysis of π-π S-wave phase shifts from π-N scattering data by the University College, London, group has produced S-wave phase shifts in harmony with the calculated phases for $\lambda\cong-0.15$ obtained in references 2 and 3. (Private communication from J. Hamilton.) See also H. J. Schnitzer, Phys. Rev. **125**, 1059 (1962).
[5] A. R. Erwin, R. March, W. D. Walker and E. West, Phys. Rev. Letters **6**, 628 (1961); E. Pickup, F. Ayer, and E. O. Salant *ibid.* **5**, 161 (1960); R. Barloutaud, J. Heughebaert, A. Levique, J. Mayer, and R. Omnès, *ibid.* **8**, 32 (1962); I. Derado and N. Schmitz, Phys. Rev. **118**, 309 (1960); J. A. Anderson, V. X. Bang, P. G. Burke, D. C. Carmony, and N. Schmitz, Phys. Rev. Letters **6**, 365 (1961); D. Stonehill, C. Baltay, H. Courant, W. Fickinger, E. C. Fowler, H. Kraybill, J. Sandweiss, and H. Taft, *ibid.* **6**, 624 (1961); V. P. Kenny, W. D. Sheppard, and C. D. Gall, Nuovo cimento **23**, 245 (1962); B. Sechi Zorn, Phys. Rev. Letters **8**, 282 (1962); C. Peck, L. Jones and M. Perl, Phys. Rev. **126**, 1836 (1962).

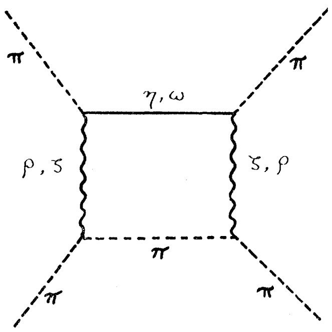


FIG. 1. Feynman graph for pion-pion scattering having a single-pion and a three-pion resonance in the intermediate state.

is[1,2]

$$\mathrm{Re}(f^{-1})=\frac{1}{a_1}-\frac{\nu-\nu_0}{\pi}P\int_0^\infty\frac{d\nu'[\nu'/(\nu'+1)]^{\frac{1}{2}}R(\nu')}{(\nu'-\nu)(\nu'-\nu_0)}+\frac{\nu-\nu_0}{\pi}P\int_1^\infty\frac{d\nu'\,\mathrm{Im}[f^{-1}(-\nu')]}{(\nu'+\nu)(\nu'+\nu_0)}-\xi_1\left(\frac{\nu-\nu_0}{\nu}\right). \quad (4)$$

When we are given σ^{in}, Eq. (4) coupled with despersion relations for the inverse S-wave amplitudes can be solved by iteration on a computer. The equations yield convergent solutions under iteration if $R(\nu)$ in (4) does not increase faster than a constant as $\nu\to\infty$. Solutions of the equations show that the nearby left-cut contributions in (4) cannot be neglected.

In Table I, the threshold ν_T for the production of pairs of particles made up of one or more resonances is shown. Since we are primarily interested in low-energy scattering ($\nu\leq 10$) only the first two cases need be considered, and these are both pure $I=1$ states. We adopt the $J=1^-$ quantum numbers for the η and ω.[6] If the η particle turns out to be $J=0^-$, this will have a small effect on the results. The inelastic amplitude is calculated from the graph shown in Fig. 1, in which the t channel consists of two di-pion resonances, each of which may be either a ρ or a ζ particle. The interaction

TABLE I. Threshold values for various two-particle approximations of the four-pion intermediate state.

Four-pion state	Threshold $=\nu_T$
$\pi+\eta$	5.1
$\pi+\omega$	10.0
$\zeta+\zeta$	15.6
$\zeta+\rho$	21.6
$\rho+\rho$	28.5

[6] A. Pevsner, R. Kraemer, M. Nussbaum, C. Richardson, P. Schlein, R. Strand, T. Toohig, M. Block, A. Engler, R. Gessaroli, and C. Meltzer, Phys. Rev. Letters **7**, 421 (1961). B. Maglić, L. Alvarez, A. Rosenfeld, and M. Stevenson, *ibid.* **7**, 178 (1961).

Hamiltonians for $\pi\pi\rho$, $\pi\rho\eta$, and $\pi\rho\omega$ are, respectively,

$$(ig'/\sqrt{2})\epsilon^{rst}\rho_\mu{}^r\partial_\mu\pi^s\pi^t,$$
$$g_\eta\epsilon_{\mu\nu\lambda\sigma}\partial_\lambda\eta_\mu\partial_\sigma\rho_\nu{}^t\pi^t, \quad (5)$$
$$g_\omega\epsilon_{\mu\nu\lambda\sigma}\partial_\lambda\omega_\mu\partial_\sigma\rho_\nu{}^t\pi^t.$$

For the ζ, if it exists and has the same quantum numbers as the ρ, the same Hamiltonians can be used with ρ replaced by ζ.

On projecting out the P wave, we find

$$\mathrm{Im}f^{\text{in}}=G^2k(\nu+1)\{a-\tfrac{1}{2}(a^2-1)\ln[(a+1)/(a-1)]\}^2, \quad (6)$$

where

$$a=(1/k\nu^{\frac{1}{2}})\{[(\nu+1)(k^2+1)]^{\frac{1}{2}}+(\rho^2-\eta^2-1)/2\},$$
$$k^2=\frac{[\nu-(\eta^2-3)/4]^2-\eta/4}{\nu+1}, \quad (7)$$

and ρ and η now denote the di-pion and three-pion masses, $G_\omega{}^2=(g_\omega g'/4\pi)^2$ and $G_\eta{}^2=(g_\eta g'/4\pi)^2$. This calculation is incorrect at high energies, since it violates unitarity. In particular this means that the contribution from the η completely dominates that of the ω. To allow for any effect the ω may have the η contribution is cut off before the $\pi+\omega$ threshold. When this cutoff is omitted, the solutions obtained are qualitatively unaltered. In order to ensure that the solutions obtained using (6) are consistent with unitarity, we expand $\mathrm{Im}f^{\text{in}}$ given in terms of scattering theory by (2) in powers of δ_I and keep only the first term

$$[\nu/(\nu+1)]^{\frac{1}{2}}\,\mathrm{Im}f^{\text{in}}\sim\delta_I. \quad (8)$$

If we assume that Eq. (8) holds everywhere, then we have a reasonable approximation for $\mathrm{Im}f^{\text{in}}$ in the low-energy region, and a solution for $\mathrm{Im}f^{\text{in}}$ which is consistent with unitarity in the entire physical region. The

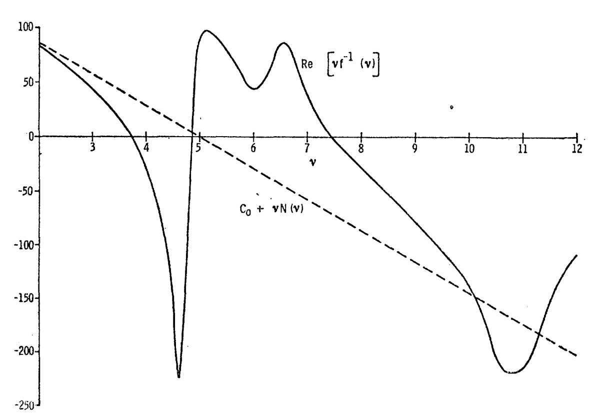
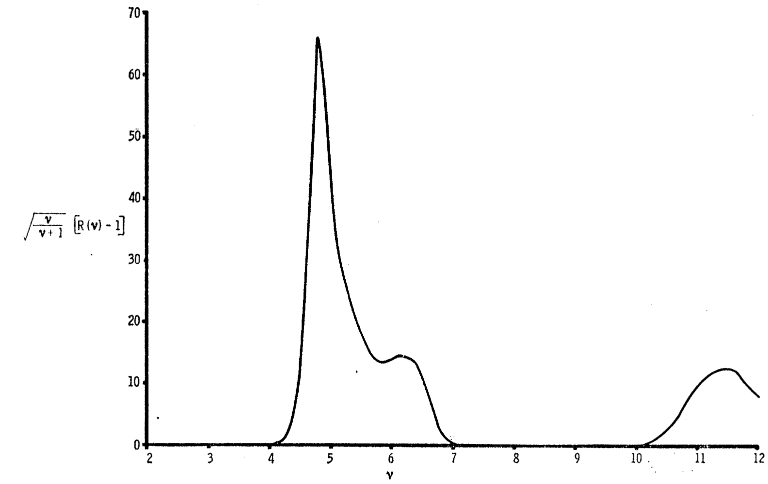


FIG. 2. The functions $\mathrm{Re}[\nu f^{-1}(\nu)]$ (solid line) and $C_0+\nu N(\nu)$ (dashed line) for the cutoff calculation. $C_0+\nu N(\nu)=\mathrm{Re}[\nu f^{-1}(\nu)]=133$ when $\nu=0$. The three zeros of $\mathrm{Re}(\nu f^{-1})$ correspond, respectively, to $\delta_R=\pi/2$, π, and $3\pi/2$.

FIG. 3. $[\nu/(\nu+1)]^{\frac{1}{2}}[R-1]$ as a function of ν for the cutoff calculation with two di-pion resonances.



equation

$$\tan 2\delta_R=\frac{2\,\mathrm{Re}f}{[(\nu+1)/\nu]^{\frac{1}{2}}-2\,\mathrm{Im}f} \tag{9}$$

is independent of δ_I and can be used to calculate δ_R giving in conjunction with (8) the result

$$R=\frac{2(1-\cos 2\delta_R e^{-2\delta_I})}{(1-2\cos 2\delta_R e^{-2\delta_I}+e^{-4\delta_I})}. \tag{10}$$

When $\delta_I=0$ it follows that $R=1$, and when $\delta_I=\infty$ we have $R=2$ corresponding to diffraction scattering. If δ_I is small and slowly varying and δ_R is close to π so that $(\delta_R-\pi)^2\ll\delta_I$, then (10) can be written

$$R\approx\delta_I/[(\delta_R-\pi)^2+\delta_I^2], \tag{11}$$

which has a resonant behavior with a maximum $R_{\max}=1/\delta_I$. The elastic cross section is given by

$$\sigma^{\mathrm{el}}\propto|f|^2/(\nu+1)=(1/\nu)(1-2\cos 2\delta_R e^{-2\delta_I}+e^{-4\delta_I}). \tag{12}$$

If δ_I is small and slowly varying, σ^{el} has a maximum at $\delta_R=(\pi/2,\,3\pi/2)$ and $\sigma_{\max}{}^{\mathrm{el}}\propto(1/\nu)(1+e^{-2\delta_I})^2$.

The following typical results have been obtained using a pion-pion coupling constant $\lambda=-0.1$, for which the S waves are in good agreement with pion production data.[3,4] For this value of λ, the P-wave constants are $a_1^{-1}=-200.0$ and $\xi_1=-199.5$.

In analyzing the results, it is instructive to write (4) in the form

$$\mathrm{Re}[\nu f^{-1}]=\nu[C_1+M+N]+C_0, \tag{13}$$

where C_1 is given by

$$C_1=a_1^{-1}-\xi_1-\frac{\nu-\nu_0}{\pi}P\int_0^\infty\frac{d\nu'[\nu'/(\nu'+1)]^{\frac{1}{2}}}{(\nu'-\nu)(\nu'-\nu_0)}, \tag{14}$$

and the inelastic contribution is

$$M=-\frac{\nu-\nu_0}{\pi}P\int_{\nu_T}^\infty\frac{[\nu'/(\nu'+1)]^{\frac{1}{2}}[R(\nu')-1]}{(\nu'-\nu)(\nu'-\nu_0)}. \tag{15}$$

The contribution from the left cut is determined by

$$N=\frac{\nu-\nu_0}{\pi}P\int_1^\infty\frac{d\nu'\,\mathrm{Im}[f^{-1}(-\nu')]}{(\nu'+\nu)(\nu'+\nu_0)}, \tag{16}$$

and $C_0=-\frac{2}{3}\xi_1=133$. The function N is slowly varying and of order -25, and $|C_1|<1$ for $0\leq\nu\leq100$. A curve of $C_0+\nu N$ is shown in Fig. 2. It is only weakly coupled to the inelastic contribution via the left cut.

First, we consider a rapidly rising σ^{in} for which the threshold $\nu_T=4.2$, and which approaches to within 7% of total absorption at $\nu=6.7$.[7–8] This corresponds to $G_\eta^2=10$ and $M_\eta=500$ MeV with no cutoff. For this σ^{in}, R rises rapidly to a maximum of 5 at $\nu=5.25$, and then falls slowly to a value of 2. The resulting inelastic contribution M has a slope which nowhere exceeds 2. This is much less than $|N|$, which is the negative slope of $C_0+\nu N$. Thus, this inelastic contribution does not change the nature of the solution.[3] The numerical results show that the position of the resonance is changed by 2%. Similar results were obtained for $G_\eta^2=0.1$ and $G_\eta^2=1$.

In the second type of solution, based on a slowly varying inelastic cross section, two resonances are found in the P-wave elastic cross section. The phase shift δ_R passes through π just above the inelastic threshold producing a large peak in R as shown in Fig. 3. This case includes a linear cutoff in σ^{in} between 740 MeV ($\nu=6$) and 790 MeV ($\nu=7$) and the values of the other parameters are $M_\eta=470$ MeV, $M_\omega=790$ MeV, and $G_\eta^2=G_\omega^2=1$. The positions of the di-pion resonances are $M_\zeta=600$ MeV and $M_\rho=800$ MeV, and the total widths are $\Gamma_\zeta=30$ MeV and $\Gamma_\rho=50$ MeV, as shown in Fig. 4. In Fig. 2 are shown $C_0+\nu N$ and $\mathrm{Re}(\nu f^{-1})$ demonstrating clearly the role played by the resonant R contribution. The first, second, and third zeros of $\mathrm{Re}(\nu f^{-1})$ correspond to $\delta_R=\pi/2$, π, and $3\pi/2$. The second and the third bumps in R which are due to the cutoff and the

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[8] R. Blankenbecler, Phys. Rev. **125**, 755 (1962).

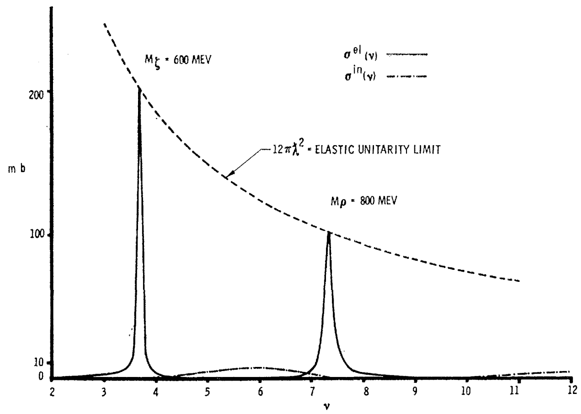

FIG. 4. The elastic and inelastic cross sections shown as full and dash-dot lines, respectively, for the cutoff calculation. The pure elastic limit is shown by the dashed line and the limit of the inelastic cross section is 1/4 of this.

$\omega+\pi$ intermediate state have little effect as can be seen in Fig. 2. It should be noted that $\mathrm{Re}(\nu f^{-1})$, $C_0+\nu N$, and νM all have zeros at or near the point where $\delta_R=\pi$; this is true in general if δ_R only passes once through π in the low-energy region.

A similar result is given by a calculation in which $\sigma^{\rm in}$ is *not* cut off. In particular, for the values of the parameters $M_\eta=475$ MeV, $G_\eta^2=1$ the two di-pion resonances occur at $M_\zeta=640$ MeV and $M_\rho=730$ MeV, and the widths are $\Gamma_\zeta=35$ MeV and $\Gamma_\rho=40$ MeV. In the latter result the contribution from the $\omega+\pi$ intermediate state is not included. The first peak in $[\nu/(\nu+1)]^{\frac{1}{2}}(R-1)$, in the case of no cutoff, has a maximum of 41 and the curve then falls rapidly almost to zero at the position of the second di-pion resonance, after which the curve rises rapidly again to about 6 and finally approaches a value corresponding to $R=2$ as $\nu\to\infty$. The theoretical maximum of $\sigma^{\rm el}$ is $\sigma_{\max}{}^{\rm el}=3\pi\bar{\lambda}^2(1+e^{-2\delta_I})^2$ and at the second resonance ($M_\rho=730$ MeV) this is 93% of the pure elastic maximum.

The position of the first resonance is very sensitive to the value of the inelastic threshold. The four-pion intermediate state threshold is, in fact, at $\nu=3$, but in the inelastic model used in the present work the threshold is that corresponding to $\eta+\pi$, and in virtue of this circumstance the η mass has been varied. All the results showing two resonances in the P wave are obtained for $M_\eta<500$ MeV. The widths of the ζ and ρ resonances are in satisfactory agreement with the experimental results, but some of our results show more than two resonances in the P wave, and it may be that the ρ consists of two narrow resonances close together.

An improved calculation of $\sigma^{\rm in}$ could be obtained by determining R from the R_{11}, R_{12}, R_{22} elements of the reaction matrix, where R_{11} is the amplitude for $\pi+\pi\leftrightarrow\pi+\pi$, R_{12} for $\pi+\pi\leftrightarrow\pi+\eta$, and R_{22} for $\pi+\eta\leftrightarrow\pi+\eta$. This calculation has the advantage of coupling $\sigma^{\rm in}$ to $\sigma^{\rm el}$.

Provided that the inelastic cross section does not vary too rapidly we conclude from these results that the inelastic intermediate states can be treated as a perturbation on the previously published solutions for elastic pion-pion scattering,[2,3] since they have little effect on the nearby left cut in the inverse amplitude which dominates the problem. These results do not depend sensitively on the model used to represent the four-pion intermediate state.

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Soft Pion Emission Induced by Electromagnetic and Weak Interactions*

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We consider some extensions of the relation which has been obtained by Nambu and Lurié between an elastic process and the accompanying soft pion emission process under the assumption of approximate γ_5 invariance. In particular, a generalized formula is found which enables one to describe the electropion production $e+N\to N+e+\pi$ and the neutrino-pion production $\nu+N\to N+e+\pi$ at the threshold of $N\pi$ system in terms of the vector (Hofstadter) and the axial-vector form factors. Explicit forms of the cross sections are given.

1. INTRODUCTION

THE possibility of approximate chirality conservation in the pion-nucleon system was first observed in connection with weak interactions, where the lepton seems to be coupled to nearly conserved axial vector as well as vector currents. The physical aspects of the conserved axial vector currents, however, are very much different from those of the vector currents associated with the ordinary conservation laws. One may regard the former as a case of hidden symmetry which does not

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