

Theory of the Fermion Liquid*

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The ground-state wave function generated by a system of interacting bosons is used as the correlation factor in a trial function describing the interacting fermion system. Expectation values are computed by the technique of the generalized normalization integral. The resulting cluster integrals are evaluated by introducing simple approximate forms for the n -particle distribution functions defined by the boson wave function. To illustrate the method, it is applied to establish a connection between known results for fermion and boson forms of a hard-sphere system at low density. As a second illustration experimental information on liquid He⁴ is used to work out properties of a hypothetical fermion-type system of mass 4. Results are given for the energy, effective mass, and magnetic behavior. The application to He³ waits on the evaluation of the radial distribution function and liquid structure factor for the corresponding artificial boson system of mass 3.

I. INTRODUCTION

A LARGE number N of identical interacting fermions of mass M are confined in a volume Ω . We are interested in the ground state and low excited states of this system in the limit $N \rightarrow \infty$, $\Omega \rightarrow \infty$, subject to the constraint $\rho = N/\Omega$ held constant. We begin by choosing a trial function ψ as a product of two factors, a completely symmetrical positive valued correlation function ψ^B and an antisymmetrical model function φ for which the simplest choice is a single Slater determinant constructed from plane wave orbitals and spin functions.¹⁻³ Thus,

$$\begin{aligned}\psi(1,2,\dots,N) &= \psi^B(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) \varphi(\mathbf{r}_1\sigma_1, \mathbf{r}_2\sigma_2, \dots, \mathbf{r}_N\sigma_N) \\ \varphi &= \sum_{\nu} (-1)^{\mu_{\nu}} P_{\nu} \varphi_0 \varphi_s, \\ \varphi_0 &= \prod_1^N e^{i\mathbf{k}_n \cdot \mathbf{r}_n}, \\ \varphi_s &= \prod_1^N S_n(\sigma_n).\end{aligned}\quad (1)$$

A cubical shape for the container Ω is convenient since it permits the explicit determination of the wave vector \mathbf{k} by a periodic boundary condition. The zero-spin ground state is characterized by a maximum wave number $k_f = (3\pi^2\rho/\nu)^{1/3}$, 2ν being the degeneracy of the spin states ($\nu=1, 2$ for He³ and nuclear matter, respectively). The spin functions $S_n(\sigma)$ are normalized and any two are either identical or orthogonal in the spin space (two orthogonal spin states for He³ and four for nuclear matter). Spin denotes either the true spin (as in He³) or both spin and isospin (as in nuclear matter).

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¹ F. Iwamoto and M. Yamada, Progr. Theoret. Phys. (Kyoto) 17, 543 (1957); 18, 345 (1957).

² C. D. Hartogh and H. A. Tolhoek, Physica 24, 721, 875, 896 (1958).

³ J. W. Clark and E. Feenberg, Phys. Rev. 113, 388 (1959).

Our first concern is the correlation factor ψ^B . This function should contain a large part of the physical effects produced by the interactions among the particles. These interactions generate strong short-range repulsive forces when two or more particles approach closely. It is plausible that correlations produced by the short-range repulsions depend little on the boundary conditions and type of permutation symmetry satisfied by the space part of the wave function. Thinking along these lines one is led to consider that the boson-type ground-state eigenfunction ψ_0^B of the Hamiltonian operator H possesses all the qualitative features demanded of a correlation factor and also includes almost all of the correlation effects generated by the short-range repulsive component in the interaction. To include both He³ and nuclear matter in a general formulation let the total Hamiltonian H be written as a sum of two terms, $H = H_0 + H_1$, with H_0 involving only space coordinates and momentum operators (in particular, the kinetic energy, all of the short-range repulsive interaction and part or all of the longer range attractive component) while H_1 contains all terms with an intrinsic dependence on spin (or isospin). The eigenvalue equation,

$$H_0\psi_0^B = E_0^B\psi_0^B, \quad (2)$$

solved for the lowest state of the boson-type spectrum at the given density ρ , generates a function ψ_0^B which is identified with ψ^B in Eq. (1). Note that in the He³ problem, H_1 contains only the spin-spin interaction and the coupling of spins to an external magnetic field.

This prescription for the correlation factor possesses practical utility, first in the sense that $E_0^B(\rho)$ and the essential probability distributions defined by ψ_0^B can be computed reasonably well and second in the sense of yielding a simple convenient formalism for computing the properties of the condensed fermion system.

Note added in proof. K. Ljolje and S. F. Wang also use ψ_0^B as the correlation factor in a trial function describing the ground state of liquid He³ (unpublished).

A detailed specification of ψ_0^B in the N -particle configuration space is, of course, impossible. However only the two- and three-particle distribution functions $p(1,2)$

and $p(1,2,3)$ are needed in the present calculation. The n -particle distribution function of a system described by a wave function ψ is defined by

$$p(1,2,\dots,N) = (N-1)(N-2)\cdots(N-n+1)\Omega \times \left(\int |\psi|^2 d\tau_{n+1\dots N} / \int |\psi|^2 d\tau_{12\dots N} \right) \quad (3)$$

the integrations over τ including summations over spin (and isospin) coordinates for fermion systems. In the special case $n=2$, we write $p=\rho g$ and obtain for the Bose system

$$\rho g_B(r_{12}) = (N-1)\Omega \int (\psi_0^B)^2 dv_{34\dots N}. \quad (4)$$

Here, the boson function ψ_0^B is taken normalized. Also needed are the liquid structure factor

$$S(k) = 1 + \rho \int e^{i\mathbf{k}\cdot\mathbf{r}} [g_B(r) - 1] dv, \quad (5)$$

and the convenient notation

$$\begin{aligned} u(k) &= S(k) - 1, \\ f(r) &= g_B(r) - 1. \end{aligned} \quad (6)$$

The actual computation of $g_B(r)$, the radial distribution function of the Bose system, from an assumed interaction between pairs of particles is a formidable task but can be carried through with the aid of extremum theorems developed and applied in a recent paper.⁴ We will return to this problem in a later paper.

II. CLUSTER DEVELOPMENT FOR THE FERMION ENERGY

The expectation value of H with respect to the trial function ψ is written

$$\langle H \rangle = \langle H_0 \rangle + \langle H_1 \rangle. \quad (7)$$

The second term $\langle H_1 \rangle$ is not considered further in the present discussion which is concerned primarily with liquid helium. However, $\langle H_1 \rangle$ is an essential element in the nuclear problem. To compute $\langle H_0 \rangle$, we make use of Eq. (2) and also the relation

$$\Delta_n \varphi_0 = -k_n^2 \varphi_0, \quad (8)$$

and obtain

$$\begin{aligned} \langle H_0 \rangle &= E_0^B + \frac{\hbar^2}{2M} \sum_n k_n^2 \\ &+ \frac{\hbar^2}{2M} \frac{\sum_n \int (\psi_0^B)^2 i\mathbf{k}_n \cdot \nabla_n (\varphi_0^* \varphi_0 \varphi_s) d\tau_{12\dots N}}{\int (\psi_0^B)^2 \varphi_0^* \varphi_0 \varphi_s d\tau_{12\dots N}}. \end{aligned} \quad (9)$$

⁴ F. Y. Wu and E. Feenberg, Phys. Rev. **122**, 739 (1961).

Direct evaluation of the integrals occurring in Eq. (9) leads to a completely useless form because the correct (linear) dependence of $\langle H_0 \rangle$ on N is hidden in the ratio of two immensely complicated polynomials in N . This difficulty is familiar in both classical and quantum theoretical treatments of many-particle problems. To avoid the difficulty, we follow the lead of Iwamoto and Yamada^{1,3} and introduce a generalized normalization integral

$$I_N(\beta) = \int (\psi_0^B)^2 \exp(i\beta \sum_\alpha \mathbf{k}_\alpha \cdot \nabla_\alpha) \varphi_0^* \varphi_0 \varphi_s d\tau_{12\dots N}. \quad (10)$$

In terms of I_N , Eq. (9) reduces to

$$\langle H_0 \rangle = E_0^B + \frac{\hbar^2}{2M} \sum_n k_n^2 + \frac{\hbar^2}{2M} \left[\frac{d}{d\beta} \ln I_N(\beta) \right]_{\beta=0}. \quad (11)$$

Thus, our problem is reduced to the evaluation of $\ln I_N(\beta)$. This is accomplished by the familiar cluster expansion formalism.⁵ Observe first that the determinantal wave function φ is conveniently expressed in terms of the exchange operators

$$\begin{aligned} \varphi &= [1 - \sum_{m < n} P_{nm}^{mn} + \sum_{l < m < n} (P_{mnl}^{lmn} + P_{nlm}^{lmn}) \\ &- \sum_{h < l < m < n} (P_{lmnh}^{hlmn} + \dots + P_{mnlh}^{hlmn}) \\ &+ P_{ml}^{lm} P_{hn}^{nh} + P_{nl}^{ln} P_{hm}^{mh} \\ &+ P_{hl}^{lh} P_{nm}^{mn}) + \dots] \varphi_0 \varphi_s. \end{aligned} \quad (12)$$

Here the operator $P_{\alpha\beta\dots\tau}^{ab\dots t}$ denotes a cyclic⁶ exchange operation which changes state a into state α [i.e., $\mathbf{k}_a \rightarrow \mathbf{k}_\alpha$, $S_a(\sigma_a) \rightarrow S_\alpha(\sigma_a)$], state b into state β , and so forth. Equation (12) gives all terms explicitly up to four exchanges. Further since any P operator corresponds to a cyclic permutation, it is clear that the expectation value of the exchange operator with respect to the spin function φ_s is either unity when all spin states are identical or zero otherwise; e.g.,

$$\begin{aligned} \langle P_{nm}^{mn} \rangle_s &= 1 \quad \text{for } S_m = S_n \\ &= 0 \quad \text{for } S_m \neq S_n. \end{aligned} \quad (13)$$

Next, observe that

$$\begin{aligned} \varphi_0 P_{nm}^{mn} \varphi_0^* &= e^{i(\mathbf{k}_m \cdot \mathbf{r}_n + \mathbf{k}_n \cdot \mathbf{r}_m)} e^{-i(\mathbf{k}_n \cdot \mathbf{r}_m + \mathbf{k}_m \cdot \mathbf{r}_n)} \\ &= e^{i\mathbf{k}_{mn} \cdot \mathbf{r}_{mn}}, \end{aligned} \quad (14)$$

$$\begin{aligned} \exp[i\beta(\mathbf{k}_m \cdot \nabla_m + \mathbf{k}_n \cdot \nabla_n)] \varphi_0 P_{nm}^{mn} \varphi_0^* \\ = e^{-\beta k_{mn}^2} e^{i\mathbf{k}_{mn} \cdot \mathbf{r}_{mn}}. \end{aligned} \quad (15)$$

⁵ B. Kahn and G. E. Uhlenbeck, Physica **5**, 399 (1938).

⁶ By cyclic we mean that the P operator cannot be separated into the product of two P operators. Thus, if one starts from any index of the set $\{a\}$ (or the set $\{\alpha\}$ since they contain the same indices) and follows the exchange procedures, one returns to the starting index after passing every other index of the set just once. Thus, P_{mnlh}^{hlmn} means $h \rightarrow m \rightarrow l \rightarrow n \rightarrow h$.

The convenient notation⁷

$$e_{ab\dots t}(1,2,\dots,n) = (-1)^{n-1} \sum e^{-(1/2)\beta(k_{ab}^2+k_{bc}^2+\dots+k_{ta}^2)} \times e^{i(k_{ab}\cdot r_1+k_{bc}\cdot r_2+\dots+k_{ta}\cdot r_n)} \langle P_{bc\dots ta}{}^{ab\dots t} \rangle_s \quad (16)$$

enables us to write

$$I_N(\beta) = \int (\psi_0^B)^2 \left\{ 1 + \sum_{m<n} e_{mn}(1,2) + \sum_{l<m<n} e_{lmn}(1,2,3) + \sum_{h<l<m<n} [e_{hlmn}(1,2,3,4) + e_{hl}(1,2)e_{mn}(3,4) + e_{hm}(1,2)e_{ln}(3,4) + e_{hn}(1,2)e_{lm}(3,4)] + \dots \right\} dv_{12\dots N}. \quad (17)$$

A sequence of cluster integrals X_{mn} , X_{lmn} , X_{hlmn} , \dots is generated by the successive approximants to $I_N(\beta)$:

$$\begin{aligned} I_{mn}(\beta) &= \int (\psi_0^B)^2 [1 + e_{mn}(1,2)] dv_{12\dots N} \\ &= 1 + X_{mn}, \\ I_{lmn}(\beta) &= \int (\psi_0^B)^2 [1 + e_{lm}(1,2) + e_{mn}(1,2) + e_{nl}(1,2) + e_{lmn}(1,2,3)] dv_{12\dots N} \\ &= 1 + X_{lm} + X_{mn} + X_{nl} + X_{lmn}, \\ I_{hlmn}(\beta) &= \int (\psi_0^B)^2 [1 + e_{hl}(1,2) + \dots + e_{mn}(1,2) + e_{hlm}(1,2,3) + \dots + e_{lmn}(1,2,3) + e_{hl}(1,2)e_{mn}(3,4) + e_{hm}(1,2)e_{ln}(3,4) + e_{hn}(1,2)e_{lm}(3,4)] dv_{12\dots N} \\ &= 1 + X_{hl} + \dots + X_{mn} + X_{hlm} + \dots + X_{lmn} + X_{hl}X_{mn} + X_{hm}X_{ln} + X_{hn}X_{lm} + X_{hlmn}. \end{aligned} \quad (18)$$

It is worthwhile stressing that the first few approximants are in no sense good approximations to I_N . However, because of the law of formation of successive approximants, the first few approximants do contain the desired information about the physical properties of the system.

Continuing in this manner, the final exact form is

$$\begin{aligned} I_{12\dots N}(\beta) &= I_N(\beta) \\ &= 1 + \sum_{m<n} X_{mn} + \sum_{l<m<n} X_{lmn} + \sum_{h<l<m<n} (X_{hl}X_{mn} + X_{hm}X_{ln} + X_{hn}X_{lm} + X_{hlmn}) + \dots + X_{12\dots N}. \end{aligned} \quad (19)$$

⁷ The summation in Eq. (16) is over the set of $(n-1)!$ cyclic permutations on the indices a, b, \dots, t . A typical term is shown.

The definitions of the first three cluster integrals are

$$\begin{aligned} X_{mn} &= \int (\psi_0^B)^2 e_{mn}(1,2) dv_{12\dots N}, \\ X_{lmn} &= \int (\psi_0^B)^2 e_{lmn}(1,2,3) dv_{12\dots N}, \\ X_{hlmn} &= \int (\psi_0^B)^2 [e_{hlmn}(1,2,3,4) + e_{hl}(1,2)e_{mn}(3,4) + e_{hm}(1,2)e_{ln}(3,4) + e_{hn}(1,2)e_{lm}(3,4)] dv_{12\dots N} \\ &\quad - X_{hl}X_{mn} - X_{hm}X_{ln} - X_{hn}X_{lm}. \end{aligned} \quad (20)$$

Equation (19) for $I_N(\beta)$ is useless for computation as it stands, since it has the same general structure (and general dependence on N) as the integrals occurring in Eq. (9). However, Eq. (11) tells us that $\ln I_N(\beta)$ is the quantity which should depend simply (i.e., linearly) on N . Iwamoto and Yamada¹ derive the formula,

$$\ln I_N(\beta) = \sum_{m<n} X_{mn} + \sum_{l<m<n} (X_{lmn} - X_{lm}X_{mn} - X_{mn}X_{nl} - X_{nl}X_{lm}) + \dots, \quad (21)$$

by a formal transformation of the cluster development for $I_N(\beta)$. A discussion and analysis of this derivation appears in Appendix A and also an alternative derivation.

Equations (11) and (21) together give

$$\begin{aligned} \langle H_0 \rangle &= E_0^B + \frac{\hbar^2}{2M} \sum_n k_n^2 + \frac{\hbar^2}{2M} \sum_{m<n} X'_{mn} \\ &\quad + \frac{\hbar^2}{2M} \sum_{l<m<n} [X'_{lmn} - X_{mn}(X'_{lm} + X'_{nl}) - X_{nl}(X'_{mn} + X'_{lm}) - X_{lm}(X'_{nl} + X'_{mn})] + \dots \end{aligned} \quad (22)$$

Here, all quantities are evaluated at $\beta=0$. The prime denotes the derivative with respect to β . This operation replaces the first exponential factor in Eq. (16) by $-\frac{1}{2}(k_{ab}^2 + \dots + k_{ta}^2)$.

The three-particle distribution function $p(1,2,3)$ is needed in the evaluation of X_{lmn} . Two approximations for $p(1,2,3)$ are available, the Kirkwood superposition form⁸

$$p_K(1,2,3) = \rho^2 g_B(r_{12}) g_B(r_{23}) g_B(r_{31}), \quad (23)$$

⁸ H. S. Green, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1960), Vol. 10, p. 52.

and the convolution form⁹

$$\begin{aligned} p_C(1,2,3) &= p_K(1,2,3) + \rho^2 \left[\rho \int f(r_{14})f(r_{24})f(r_{34})dv_4 \right. \\ &\quad \left. - f(r_{12})f(r_{23})f(r_{31}) \right] \\ &= \rho^2 \left[1 + f(r_{12}) + f(r_{23}) + f(r_{31}) \right. \\ &\quad \left. + f(r_{12})f(r_{23}) + f(r_{23})f(r_{31}) + f(r_{31})f(r_{12}) \right. \\ &\quad \left. + \rho \int f(r_{14})f(r_{24})f(r_{34})dv_4 \right]. \quad (24) \end{aligned}$$

Only p_K is suitable for computing matrix elements of singular operators. However, p_C has two advantages over p_K which recommend its use in computing matrix elements of nonsingular operators. These advantages are: (a) p_C satisfies exactly the sequential relation connecting two- and three-particle distribution functions,⁹ and (b) matrix elements computed using p_C are easily evaluated. The extent to which matrix elements computed using p_K or p_C are in error is not known, nor when the operator is nonsingular, which form gives more accurate results.

No approximation is required to evaluate the two index cluster integral. Equations (5), (6), (16), and (20) yield

$$X_{mn} = -(1/N)e^{-\beta k_{mn}^2} u(k_{mn}) \langle P_{nm}^{mn} \rangle_s. \quad (25)$$

For the three index integrals we use the convolution form for $p(1,2,3)$ and obtain

$$\begin{aligned} X_{lmn} &= (2/N^2)e^{-(1/2)\beta(k_{lm}^2 + k_{mn}^2 + k_{nl}^2)} [u(k_{lm})u(k_{ln}) \\ &\quad + u(k_{mn})u(k_{ml}) + u(k_{nl})u(k_{nm}) \\ &\quad + u(k_{lm})u(k_{mn})u(k_{nl})] \langle P_{mnl}^{lmn} \rangle_s. \quad (26) \end{aligned}$$

With the help of Eqs. (25) and (26), Eq. (22) for the expectation value of H_0 can be reduced to the explicit form

$$\begin{aligned} \langle H_0 \rangle &= E^F = E_0^B + \Delta E^F, \\ \Delta E^F &= E_1^F + E_2^F + E_3^F, \end{aligned} \quad (27)$$

in which terms of order higher than those given in Eq. (22) are neglected, and

$$\begin{aligned} E_1^F &= \frac{\hbar^2}{2M} \sum_n k_n^2, \\ E_2^F &= \frac{\hbar^2}{2MN} \sum_{m<n} k_{mn}^2 u(k_{mn}) \langle P_{nm}^{mn} \rangle_s, \\ E_3^F &= -\frac{3\hbar^2}{2MN^2} \sum_{l<m<n} k_{lm}^2 S(k_{lm}) u(k_{mn}) u(k_{nl}) \langle P_{mnl}^{lmn} \rangle_s. \end{aligned} \quad (28)$$

⁹ H. W. Jackson and E. Feenberg, Revs. Modern Phys. (to be published).

In the ground state the momentum distributions are expected to fill a Fermi sphere of radius $\hbar k_F$ for each spin (and isospin) orientation. Let e_F denote the maximum kinetic energy in a degenerate gas of free fermions; i.e.,

$$e_F = \hbar^2 k_F^2 / 2M = (\hbar^2 / 2M) (3\pi^2 \rho / \nu)^{2/3}. \quad (29)$$

The prescription for the expectation values of the spin permutation operators [Eq. (13)] in combination with the condition of filled Fermi spheres reduces the energy formula [Eqs. (27) and (28)] to the explicit form

$$\begin{aligned} E_0^F &= E_0^B + E_{01}^F + E_{02}^F + E_{03}^F, \\ E_{01}^F &= \frac{2}{3} e_F N, \\ E_{02}^F &= \frac{e_F N}{\nu} \left(\frac{3}{8\pi} \right)^2 \int x_{12}^2 u(k_F x_{12}) d\mathbf{x}_1 d\mathbf{x}_2 \\ &= -\frac{24}{\nu} e_F N \int_0^1 u(2k_F x) \left(1 - \frac{3}{2}x + \frac{1}{2}x^3 \right) x^4 dx, \\ E_{03}^F &= -\frac{e_F N}{\nu^2} \left(\frac{3}{8\pi} \right)^3 \int x_{12}^2 S(k_F x_{12}) u(k_F x_{23}) \\ &\quad \times u(k_F x_{31}) d\mathbf{x}_1 d\mathbf{x}_2 d\mathbf{x}_3. \end{aligned} \quad (30)$$

In the integrals each variable \mathbf{x}_n ranges over a unit sphere. Also $\mathbf{x}_1 = \mathbf{k}_1 / k_F$, etc.

The equilibrium density depends slightly on the statistics. To compute the equilibrium density ρ_F of the fermion system at $P = T = 0$ observe that [for $(\rho - \rho_B)^2 \ll \rho_B^2$]

$$E_0^B(\rho) = E_0^B(\rho_B) + \frac{1}{2} N M c_B^2 [(\Omega - \Omega_B) / \Omega_B]^2. \quad (31)$$

Here, we use the relations

$$\Omega^2 (\partial^2 E / \partial \Omega^2) = N / \rho K = N M c^2 \quad (32)$$

connecting the second derivative of the energy with respect to volume with the compressibility $K(\rho)$ and the velocity of first sound $c(\rho)$ at absolute zero. The corresponding formulas for $E^F(\rho)$ require evaluation of first and second derivatives of E_{0n}^F with respect to ρ (or Ω). The leading term E_{01}^F presents no difficulty, but the others require knowledge of how $S(k)$ depends on ρ . To estimate the first derivative, we introduce a scale transformation $\mathbf{r} \rightarrow \lambda \mathbf{r}$ into $g_B(\mathbf{r})$ to adapt the observed function to a range of densities. The normalization condition,

$$\rho \int [g_B(\mathbf{r}) - 1] d\mathbf{v} = -1, \quad (33)$$

requires $\rho / \lambda^3 = \rho_B$ or $\lambda = (\rho / \rho_B)^{1/3}$. The corresponding transformation for k is $k \rightarrow k / \lambda$. Observe now that $S(k_F(\rho_B / \rho)^{1/3} x)$ is independent of ρ . Consequently, all successive approximants [as in Eq. (30)] have the same dependence on ρ (varying as $\rho^{2/3}$).

This procedure may give a reasonable estimate for the first derivative of $\Delta E^F(\rho)$ with respect to the volume, but cannot be trusted for computing higher derivatives. In particular, experience with related problems teaches that the second derivative may be smaller by an order of magnitude than the estimate derived from the expectation value of H_0 with respect to the scaled wave function.

These remarks serve to introduce formulas exhibiting the dependence (in a schematic manner) of E^F and related quantities on ρ (or Ω). The notation

$$\begin{aligned}\delta' &= -\left[\Omega \frac{\partial \Delta E^F}{\partial \Omega} \bigg/ \frac{2}{3} \Delta E^F \right]_{\rho_B}, \\ \delta'' &= \left[\Omega^2 \frac{\partial^2 \Delta E^F}{\partial \Omega^2} \bigg/ \frac{10}{9} \Delta E^F \right]_{\rho_B},\end{aligned}\quad (34)$$

concentrates the uncertainty in the derivatives in dimensionless parameters δ' and δ'' . From the preceding discussion, we consider $\delta' \sim 1$ and $\delta'' \ll 1$ as reasonable possibilities.

In terms of δ' and δ'' ,

$$\begin{aligned}E^F(\rho) &= E^F(\rho_B) - \frac{2}{3} \Delta E^F(\rho_B) \delta' \frac{\Omega - \Omega_B}{\Omega_B} \\ &+ \frac{1}{2} N M c_B^2 \left(1 + \delta'' \frac{10}{9} \frac{\Delta E^F(\rho_B)}{N M c_B^2} \right) \left(\frac{\Omega - \Omega_B}{\Omega_B} \right)^2,\end{aligned}\quad (35)$$

$$(\Omega_F - \Omega_B)/\Omega_B \sim \delta' \frac{2}{3} \Delta E^F(\rho_B)/N M c_B^2. \quad (36)$$

It is clear that an external pressure is required to maintain $\rho = \rho_B$ when the statistics is changed; consequently δ' is certainly positive. The correction term δ'' in Eq. (35) accounts for the decreased compressibility associated with the zero-point motion of the degenerate fermion gas as compared with the corresponding boson gas.

III. CLUSTER DEVELOPMENT FOR THE FERMION RADIAL DISTRIBUTION FUNCTION

We wish to express the fermion radial distribution function in the form

$$g_F(r) = g_B(r) F(r), \quad (37)$$

with $F(r)$ given by a practically useful cluster development. To compute $g_F(r)$, we start from the observation that for any function $V(r)$

$$\left\langle \sum_{i < j} V(r_{ij}) \right\rangle = \frac{1}{2} N \rho \int g_F(r) V(r) dv. \quad (38)$$

Thus, $g_F(r)$ is the coefficient of $\frac{1}{2} N \rho V(r)$ in the integral formula for the expectation value of $\sum_{i < j} V(r_{ij})$.

To compute the expectation value in Eq. (38), we

introduce a second generalized normalization integral

$$J_N(\alpha) = \int (\psi_0^B)^2 \varphi^* \exp\left[\alpha \sum_{i < j} V(r_{ij})\right] \varphi_0 \varphi_s d\tau_{12} \dots N, \quad (39)$$

with the property

$$\left\langle \sum_{i < j} V(r_{ij}) \right\rangle = \left[\frac{d}{d\alpha} \ln J_N(\alpha) \right]_{\alpha=0}, \quad (40)$$

and compute $J_N(\alpha)$ by a sequence of successive approximations paralleling the development in Sec. II:

$$\begin{aligned}J_{mn}(\alpha) &= \int (\psi_0^B)^2 e^{\alpha V(r_{12})} [1 + e_{mn}(1,2)] dv_{12} \dots N \\ &= 1 + Y_{mn}, \\ J_{lmn}(\alpha) &= \int (\psi_0^B)^2 \exp[\alpha (V(r_{12}) + V(r_{23}) + V(r_{31}))] \\ &\quad \times [1 + e_{lm}(1,2) + e_{mn}(1,2) \\ &\quad + e_{nl}(1,2) + e_{lmn}(1,2,3)] dv_{12} \dots N \\ &= 1 + Y_{lm} + Y_{mn} + Y_{nl} + Y_{lmn}.\end{aligned}\quad (41)$$

The exact parallels to Eqs. (19) and (21) now hold for $J_N(\alpha)$ with Y cluster integrals substituted for the corresponding X 's. We need explicit formulas for the Y -type cluster integrals and their derivatives with respect to α (denoted by Y') all evaluated at $\alpha=0$. Equation (41) yields (at $\alpha=0$)

$$\begin{aligned}Y_{mn} &= - \int (\psi_0^B)^2 e^{i \mathbf{k}_{mn} \cdot \mathbf{r}_{12}} dv_{12} \dots N \langle P_{nm}^{mn} \rangle_s, \\ Y'_{mn} &= \int (\psi_0^B)^2 V(r_{12}) [1 - e^{i \mathbf{k}_{mn} \cdot \mathbf{r}_{12}} \langle P_{nm}^{mn} \rangle_s] dv_{12} \dots N, \\ Y_{lmn} &= 2 \int (\psi_0^B)^2 e^{i(\mathbf{k}_{lm} \cdot \mathbf{r}_{12} + \mathbf{k}_{mn} \cdot \mathbf{r}_{23} + \mathbf{k}_{nl} \cdot \mathbf{r}_{31})} dv_{12} \dots N \langle P_{lmn}^{lmn} \rangle_s, \\ Y'_{lmn} &= - \int (\psi_0^B)^2 [V(r_{12}) + V(r_{23}) + V(r_{31})] \\ &\quad \times [e^{i \mathbf{k}_{lm} \cdot \mathbf{r}_{12}} \langle P_{nm}^{mn} \rangle_s + e^{i \mathbf{k}_{nl} \cdot \mathbf{r}_{12}} \langle P_{ln}^{nl} \rangle_s] dv_{12} \dots N \\ &\quad + 2 \int (\psi_0^B)^2 [V(r_{12}) + V(r_{23}) + V(r_{31})] \\ &\quad \times e^{i(\mathbf{k}_{lm} \cdot \mathbf{r}_{12} + \mathbf{k}_{mn} \cdot \mathbf{r}_{23} + \mathbf{k}_{nl} \cdot \mathbf{r}_{31})} dv_{12} \dots N \langle P_{lmn}^{lmn} \rangle_s.\end{aligned}\quad (42)$$

The statement of results is simplified by introducing the function

$$\begin{aligned}h(k_F r) &= \frac{2\nu}{N} \sum_{n=1}^{N/2\nu} e^{i \mathbf{k}_n \cdot \mathbf{r}} \\ &= 3(\sin k_F r - k_F r \cos k_F r)/(k_F r)^3,\end{aligned}\quad (43)$$

appropriate for equal populations in all spin and isospin states. Equations (38), (40), (42), and (43) in conjunction with the Y -type analog of Eq. (21) and using the Kirkwood form for $p(1,2,3)$ now yield

$$g_F(r_{12}) = g_B(r_{12}) \left[1 - \frac{1}{2\nu} \hbar^2 (k_F r_{12}) \right. \\ \left. - \frac{\rho}{\nu} \int g_B(r_{23}) f(r_{13}) \hbar^2 (k_F r_{23}) dv_3 \right. \\ \left. + \frac{\rho}{2\nu^2} \hbar (k_F r_{12}) \int g_B(r_{23}) f(r_{13}) \right. \\ \left. \times \hbar (k_F r_{13}) \hbar (k_F r_{23}) dv_3 + \dots \right]. \quad (44)$$

Other two-particle distribution functions can be defined by associating projection operators Q_{ij} with the potential $V(r_{ij})$; i.e., operators to select out pairs of particles for which the spin (and isospin) state is prescribed. Such functions should prove useful in studying various aspects of the nuclear problem.²

IV. CONNECTION WITH A QUASI-PARTICLE FORMALISM

So far we have computed diagonal matrix elements of the identity and of the Hamiltonian operator using basis functions generated by single-model configurations. Observe that the basis functions [Eq. (1)] do not form an orthogonal set and obviously are not eigenfunctions of H . Nondiagonal matrix elements are needed to improve the theory of the ground state and to determine the spectrum of excitations involved in the properties of the liquid at low temperatures. In the absence of information on nondiagonal elements, what physical consequences can be deduced from the set of diagonal elements? First of all, the ground-state energy and equilibrium density are probably determined with moderate accuracy by the single ground-state basis function. The same remark applies to the exchange integrals involved in the magnetic properties of the ground state (although the accuracy may not be good enough to distinguish between strong paramagnetism and ferromagnetism). The velocity of ordinary sound is also given with fair accuracy simply because the elastic properties of the ground state are largely determined by the boson energy term $E_0^B(\rho)$.

If we leave the ground state and ask for the level density (or the effective mass) involved in computing the specific heat, the situation is much less favorable. Just the fact that the basis functions do not form an orthogonal set destroys the possibility of giving a rigorous interpretation to the spectrum of expectation values. Nevertheless, it is worthwhile for the purpose of orientation to use the expectation values to compute an admittedly crude approximation to the true energy

spectrum. This objective is attained by using Eqs. (27) and (28) to define quasi-particle energies in the manner of Landau's phenomenological theory of liquid He^3 .^{10,11}

For spin $\frac{1}{2}$ particles, let $n_{\pm}(\mathbf{k})$ denote the occupation number of the single-particle orbital defined by the wave vector \mathbf{k} and the spin orientation $+$ or $-$. These numbers can be combined to form the distribution function $n(\mathbf{k}, \sigma)$ expressed as a diagonal matrix in the spin space by the formulas

$$n(\mathbf{k}, \sigma) = \bar{n}(\mathbf{k}) + \Delta n(\mathbf{k}) \sigma_z, \\ \bar{n}(\mathbf{k}) = \frac{1}{2} [n_+(\mathbf{k}) + n_-(\mathbf{k})], \\ \Delta n(\mathbf{k}) = \frac{1}{2} [n_+(\mathbf{k}) - n_-(\mathbf{k})]. \quad (45)$$

The implicit summation over spin states in the preceding analysis is now replaced by an explicit trace notation using the symbol T_n to denote the diagonal sum over spin states $+$ or $-$ of the n th particle. Also, the spin exchange operators are given by

$$P_{21}^{12} = \frac{1}{2} (1 + \sigma_1 \cdot \sigma_2), \\ P_{231}^{123} = P_{31}^{13} P_{21}^{12}. \quad (46)$$

The following notation proves convenient:

$$K_{12} = (\hbar^2 k_{12}^2 / 2M) u(k_{12}), \\ K_{12;3} = (\hbar^2 k_{12}^2 / 2M) S(k_{12}) u(k_{23}) u(k_{31}), \\ n_1 = n(\mathbf{k}_1, \sigma_1), \quad n_{1+} = n_+(\mathbf{k}_1), \text{ etc.} \quad (47)$$

Using Eqs. (45), (46), and (47), the several terms occurring in Eqs. (27) and (28) can be written in a form suitable for arbitrary distribution of particles in the single-particle states:

$$E_1^F = \frac{\Omega}{(2\pi)^3} \int \frac{\hbar^2 k^2}{2M} T n(\mathbf{k}) d\mathbf{k} \\ = \frac{N}{(2\pi)^3 \rho} \int \frac{\hbar^2 k^2}{2M} (n_+ + n_-) d\mathbf{k}, \\ E_2^F = \frac{\Omega^2}{2N(2\pi)^6} \int K_{12} T_1 T_2 (P_{21}^{12} n_1 n_2) d\mathbf{k}_1 d\mathbf{k}_2 \\ = \frac{N}{2(2\pi)^6 \rho^2} \int K_{12} (n_{1+} n_{2+} + n_{1-} n_{2-}) d\mathbf{k}_1 d\mathbf{k}_2, \\ E_3^F = - \frac{\Omega^3}{2N^2(2\pi)^9} \int K_{12;3} T_1 T_2 T_3 (P_{231}^{123} n_1 n_2 n_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 \\ = - \frac{N}{2(2\pi)^9 \rho^3} \\ \times \int K_{12;3} (n_{1+} n_{2+} n_{3+} + n_{1-} n_{2-} n_{3-}) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \quad (48)$$

¹⁰ L. D. Landau, Soviet Phys.—JETP **3**, 920 (1957); **5**, 101 (1957).

¹¹ A. A. Abrikosov and I. M. Khalatnikov, *Reports on Progress in Physics* (The Physical Society, London, 1959), Vol. 22, p. 329; Soviet Phys.—Uspekhi **1**, 68 (1958).

A variation δE^F in the total energy is generated by the replacement $n \rightarrow n + \delta n$. Landau defines the quasi-particle energy $e(\mathbf{k}, \boldsymbol{\sigma})$ as the coefficient of $\delta n(\mathbf{k}, \boldsymbol{\sigma})$ in the formula for δE^F :

$$\delta E^F = \frac{\Omega}{(2\pi)^3} T \int e(\mathbf{k}, \boldsymbol{\sigma}) \delta n(\mathbf{k}, \boldsymbol{\sigma}) d\mathbf{k}. \quad (49)$$

The explicit development of δE^F from Eqs. (27) and (48) yields

$$\begin{aligned} e(\mathbf{k}_1, \boldsymbol{\sigma}_1) = & \frac{\hbar^2 k_1^2}{2M} + \frac{1}{(2\pi)^3 \rho} \int K_{12} T_2 (P_{21}^{12} n_2) d\mathbf{k}_2 \\ & - \frac{1}{2(2\pi)^6 \rho^2} \int (K_{12;3} + K_{23;1} + K_{31;2}) \\ & \times T_2 T_3 (P_{231}^{123} n_2 n_3) d\mathbf{k}_2 d\mathbf{k}_3. \end{aligned} \quad (50)$$

The spin dependence of $e(\mathbf{k}_1, \boldsymbol{\sigma}_1)$ is made explicit by introducing

$$\begin{aligned} T_2 (P_{21}^{12} n_2) = & \bar{n}_2 + \sigma_{1z} \Delta n_2, \\ T_2 T_3 (P_{231}^{123} n_2 n_3) = & \frac{1}{2} (n_{2+} n_{3+} + n_{2-} n_{3-}) \\ & + \frac{1}{2} (n_{2+} n_{3+} - n_{2-} n_{3-}) \sigma_{1z}. \end{aligned} \quad (51)$$

For the zero-spin state distribution (completely filled Fermi spheres) $e(\mathbf{k}_1, \boldsymbol{\sigma}_1)$ is independent of spin and

$$\begin{aligned} e(k_1)/e_F = & x_1^2 + \frac{3}{8\pi} \int x_{12}^2 u(k_F x_{12}) d\mathbf{x}_2 \\ & - \frac{1}{2} \left(\frac{3}{8\pi} \right)^2 \int [2x_{12}^2 S(k_F x_{12}) u(k_F x_{23}) \\ & + x_{23}^2 S(k_F x_{23}) u(k_F x_{12})] u(k_F x_{13}) d\mathbf{x}_2 d\mathbf{x}_3. \end{aligned} \quad (52)$$

The \mathbf{x} integrations extend over the unit sphere, as in Eq. (30).

The variation of $e(\mathbf{k}_1, \boldsymbol{\sigma}_1)$ when $n(\mathbf{k}, \boldsymbol{\sigma})$ is varied has the form

$$\delta e(\mathbf{k}_1, \boldsymbol{\sigma}_1) = \frac{\Omega}{(2\pi)^3} T_2 \int f(\mathbf{k}_1, \boldsymbol{\sigma}_1; \mathbf{k}_2, \boldsymbol{\sigma}_2) \delta n(\mathbf{k}_2, \boldsymbol{\sigma}_2) d\mathbf{k}_2, \quad (53)$$

in which $f(\mathbf{k}_1, \boldsymbol{\sigma}_1; \mathbf{k}_2, \boldsymbol{\sigma}_2)$ is a symmetrical function of $\mathbf{k}_1, \boldsymbol{\sigma}_1$ and $\mathbf{k}_2, \boldsymbol{\sigma}_2$. Equation (50) yields

$$\begin{aligned} N f(\mathbf{k}_1, \boldsymbol{\sigma}_1; \mathbf{k}_2, \boldsymbol{\sigma}_2) = & \left[K_{12} - \frac{1}{2(2\pi)^3 \rho} \int (K_{12;3} + K_{23;1} + K_{31;2}) \right. \\ & \times T_3 (P_{31}^{13} + P_{23}^{32}) n_3 d\mathbf{k}_3 \left. \right] P_{21}^{12} \\ = & \left[K_{12} - \frac{1}{(2\pi)^3 \rho} \int (K_{12;3} + K_{23;1} + K_{31;2}) \right. \\ & \times [\bar{n}_3 + \frac{1}{2} \Delta n_3 (\sigma_{1z} + \sigma_{2z})] d\mathbf{k}_3 \left. \right] \frac{1}{2} (1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2). \end{aligned} \quad (54)$$

Terms of order of $(\Delta n)^2$ may be neglected leaving $f(\mathbf{k}_1, \boldsymbol{\sigma}_1; \mathbf{k}_2, \boldsymbol{\sigma}_2)$ in the form

$$f(\mathbf{k}_1, \boldsymbol{\sigma}_1; \mathbf{k}_2, \boldsymbol{\sigma}_2) = f_0(\mathbf{k}_1, \mathbf{k}_2) (1 + \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2), \quad (55)$$

with

$$\begin{aligned} N f_0(\mathbf{k}_1, \mathbf{k}_2) = & \frac{1}{2} K_{12} - \frac{1}{2(2\pi)^3 \rho} \\ & \times \int (K_{12;3} + K_{23;1} + K_{31;2}) \bar{n}_3 d\mathbf{k}_3. \end{aligned} \quad (56)$$

In the application to the zero-spin ground-state distribution, we have

$$\begin{aligned} \frac{N}{e_F} f_0(\mathbf{k}_1, \mathbf{k}_2) = & \frac{1}{2} x_{12}^2 u(k_F x_{12}) - \frac{3}{16\pi} \int [x_{12}^2 S(k_F x_{12}) u(k_F x_{13}) u(k_F x_{23}) \\ & + x_{23}^2 S(k_F x_{23}) u(k_F x_{21}) u(k_F x_{31}) \\ & + x_{31}^2 S(k_F x_{31}) u(k_F x_{23}) u(k_F x_{12})] d\mathbf{x}_3, \end{aligned} \quad (57)$$

with \mathbf{x} integrations extending over the unit sphere. Further, $f_0(\mathbf{k}_1, \mathbf{k}_2)$ is required only for $k_1 = k_2 = k_F$; under this condition let

$$f_{00}(k_F, \cos(\mathbf{k}_1, \mathbf{k}_2)) \equiv f_0(\mathbf{k}_1, \mathbf{k}_2)_{k_1=k_2=k_F}. \quad (58)$$

This completes the identification of the basic quantities in the phenomenological description of the fermion system.

The effective mass M^* , magnetic susceptibility χ , and velocity of first sound c_F are derived quantities given by the following formulas¹¹:

$$\frac{M}{M^*} = 1 - \frac{3N}{4e_F} \int_{-1}^1 f_{00}(k_F, y) y dy, \quad (59)$$

$$\begin{aligned} \frac{\chi_0}{\chi} = & \frac{M}{M^*} + \frac{3N}{4e_F} \int_{-1}^1 f_{00}(k_F, y) dy \\ = & 1 + \frac{3N}{4e_F} \int_{-1}^1 f_{00}(k_F, y) (1 - y) dy, \end{aligned} \quad (60)$$

$$\chi_0 = \frac{1}{\pi} \left(\frac{3}{\pi} \right)^{1/3} \frac{\mu^2 M \rho^{1/3}}{\hbar^2} = \frac{\rho \mu^2}{(2/3) e_F}, \quad (61)$$

$$c_F(\rho) = c_B(\rho) \left[1 + \frac{1}{3} \left(\frac{\hbar k_F}{M c_B} \right)^2 \frac{\chi_0}{\chi} \right]^{1/2}. \quad (62)$$

In Eq. (61), μ is the magnetic moment of the He^3 nucleus and χ_0 is the susceptibility (per unit volume) of a degenerate gas of noninteracting fermions. Our formula for c_F takes account of the fact that the level from which the chemical potential is measured includes a density dependent term E_0^B/N . This adds a term c_B^2

TABLE I. Measured and computed properties of boson and fermion systems (Mass 3 and 4) at $T=0$. First and fourth rows: experimental values at $P=0^*$; second and third rows: computed values at $\rho=\rho_B$.

Mass	Statistics	Spin	(\AA^{-3})	$(^\circ\text{K})$	K (atm $^{-1}$)	c (m/sec)	E_0/N ($^\circ\text{K}$)	M^*/M	χ/χ_0
4	BE	...	0.0218	...	0.0122	239	-7.14
4	FD	0	0.0218	4.52	c	c	-5.45	1.21	c
4	FD	$\pm\frac{1}{2}N$	0.0218	7.37	-5.62	...	d
3	FD	0	0.0164	4.96	0.038	180	-2.52	2.82 ^b	11.1

* K. R. Atkins, *Liquid Helium* (Cambridge University Press, New York, 1959).

^b A. C. Anderson, G. L. Salinger, W. A. Steyert, and J. C. Wheatley, *Phys. Rev. Letters* **7**, 295 (1961).

^c The present calculation may not give satisfactory numbers because either (1) density dependence of $S(k)$ is required [Eqs. (62), (69)]; or (2) the n -particle cluster integrals ($n>3$) cannot be neglected [Eqs. (60), (62)].

^d Ferromagnetic ground state.

to the formula given in reference 11 for c_F^2 . Note that all quantities in Eq. (62) are evaluated at the same density.

Equation (62) reveals a curious connection between the susceptibility and the velocity of sound. This correlation of elastic and magnetic properties can be traced to the fact that in Eq. (55) only one function occurs, the common factor of "1" and " $\sigma_1 \cdot \sigma_2$ ". In Landau's formulation the analog of Eq. (55) is

$$f(\mathbf{k}_1, \mathbf{k}_2) + \zeta(\mathbf{k}_1, \mathbf{k}_2) \sigma_1 \cdot \sigma_2, \quad (63)$$

with no connection between f and ζ provided by the general phenomenological considerations. In the present context, the difference between f and ζ comes from cluster integrals involving four or more particles.

It is possible to derive alternative expressions for the physical quantities of interest. For example, the effective mass is given directly by the simple formula

$$M^* = \hbar^2 k_F / [de(k)/dk]_{k=k_F}. \quad (64)$$

To compute the susceptibility we observe that the numbers N_+ and N_- of particles with + and - spin orientations differ from $\frac{1}{2}N$ when the system is in a magnetic field. With $N_{\pm} = \frac{1}{2}N(1 \pm x)$, the ground-state energy in the magnetic field \mathcal{H} has the form

$$E_0^F(\rho, x) = E_0^B(\rho) + V(\rho, x) + V(\rho, -x) - N\mu\mathcal{H}x \\ \cong E_0^B(\rho) + 2V(\rho, 0) + x^2 V''(\rho, 0) - N\mu\mathcal{H}x, \quad (65)$$

in which

$$V''(\rho, 0) = [\partial^2 V(\rho, x) / \partial x^2]_{x=0}.$$

Equations (30) and (48) yield

$$2V(\rho, x) = E_{01}^F(\rho, x)(1+x) \\ + E_{02}^F(\rho, x)(1+x)^2 + E_{03}^F(\rho, x)(1+x)^3. \quad (66)$$

In $E_{0n}^F(\rho, x)$ the first variable ρ enters because of the dependence of $S(k)$ on density. The second variable x results from the replacement of k_F by $k_F^+ = k_F(1+x)^{1/3}$ in Eq. (30).

The condition for minimum energy [if $V''(\rho, 0) > 0$],

$$x = N\mu\mathcal{H} / 2V''(\rho, 0), \quad (67)$$

has the consequence that

$$\chi \cong N\mu x / \Omega\mathcal{H} \\ = N\rho\mu^2 / 2V''(\rho, 0). \quad (68)$$

A general proof that the alternative formulas for M^* are equivalent has been constructed for the E_{01}^F and E_{02}^F terms of Eq. (66). The complete identity of the alternate expressions is verified in Sec. V for a particular choice of $S(k)$.

Equations (32) and (35) provide another expression for the velocity of first sound

$$c_F(\rho_B) = c_B(\rho_B) \left[1 + \frac{10}{9} \frac{\Delta E^F}{NM c_B^2} \delta'' \right]^{1/2}, \quad (69)$$

which should reproduce the result of Eq. (62) if the density dependence of $S(k)$ is taken into account correctly. We have verified the connection for a dilute system of hard spheres [Eqs. (80) and (82)].

V. A NUMERICAL EXAMPLE

To learn something about the magnitudes of the quantities occurring in the theory we have made calculations for a hypothetical helium type fermion system (spin $\frac{1}{2}$) of mass 4 (at the experimental density of He II). For $S(k)$ we use the empirical function obtained from the analysis of x-ray diffraction in liquid He II at essentially zero pressure and temperature.¹² This function occupies a central position in recent theoretical studies of the ground state and low excited states of liquid He II.^{4,9,13} The equilibrium density ρ_B and the binding energy per particle $E_0^B(\rho_B)/N$ are also needed. Measured values appear in the first and fourth rows of Table I. Values for He³ are given to show the relative importance of the dependence on mass and statistics.

The needed integrals are evaluated in Appendix B. In the following formulas B has the value 0.195. Equations (30), (B3), and (B4) (with $\nu=1$) lead to

$$\frac{1}{N} E_0^F(\rho_B) = \frac{1}{N} E_0^B(\rho_B) + \frac{3}{5} e_F \left[1 - \left(\frac{1}{2} - \frac{6}{7} B \right) \right. \\ \left. - \left(\frac{3}{7} B - \frac{352}{315} B^2 + \frac{472}{735} B^3 \right) \right] \\ = -7.14 + 2.72(1 - 0.333 - 0.046)^\circ\text{K} \\ = -5.45^\circ\text{K}. \quad (70)$$

¹² The data on $S(k)$ is taken from L. Goldstein and J. Reekie, *Phys. Rev.* **98**, 857 (1955) and modified as described in reference 4.

¹³ H. W. Jackson and E. Feenberg, *Ann. Phys. (New York)* **15**, 266 (1961).

for the state of zero spin (equal populations of the $+$ and $-$ Fermi spheres). Here each term E_{0n}^F is evaluated separately to exhibit the convergence of the cluster formula. The convergence appears to be good, perhaps deceptively so. Obviously, E_0^B and probably also ρ_B serve very well as physically meaningful zeroth approximations in the description of the fermion system. This result conforms to the conclusion reached by de Boer¹⁴ in his analysis of the quantum mechanical principle of corresponding states (internal energy and specific volume depend primarily on the magnitude of the strength parameter Λ^* and relatively little on the statistics).

The combination of Eqs. (59), (60), and (B8) yields

$$\begin{aligned}\frac{M}{M^*} &= 1 - \left(\frac{1}{2} - 2B\right) - \left(B - \frac{148}{35}B^2 + \frac{1844}{525}B^3\right) + \dots \\ &= 1 - 0.110 - 0.060 + \dots \\ &= 1/1.21 + \dots,\end{aligned}\quad (71)$$

$$\begin{aligned}\frac{\chi_0}{\chi} &= 1 - (2 - 6B) - \left(\frac{39}{7}B - \frac{704}{35}B^2 + \frac{8024}{525}B^3\right) + \dots \\ &= 1 - 0.830 - 0.435 + \dots.\end{aligned}\quad (72)$$

The negative value computed for χ can be traced back to a negative value for $V''(\rho, 0)$ [Eqs. (65) and (66)]. This means that we have determined x at a maximum value of the energy and not at a minimum as required by the physical meaning of the susceptibility.

With no external field minimum energy occurs at (or near) $x = \pm 1$ with the value

$$\begin{aligned}\frac{1}{N}E_0^F(\rho_B, x = \pm 1) &= \frac{1}{N}E_0^B(\rho_B) + \frac{1}{N}V(\rho, 1) \\ &= \frac{1}{N}E_0^B(\rho_B) + \frac{3}{5}2^{2/3}e_F(\rho_B) \left[1 - 2\left(\frac{1}{2} - \frac{6}{7}2^{2/3}B\right) \right. \\ &\quad \left. - 4\left(\frac{3}{7}2^{2/3}B - \frac{352}{315}2^{4/3}B^2 + \frac{472}{735}2^{2/3}B^3\right) + \dots \right] \\ &= -7.14 + 2.72(1.587 - 0.745 - 0.283 + \dots)^\circ\text{K} \\ &= -5.62^\circ\text{K},\end{aligned}\quad (73)$$

lower by 0.17°K than the lowest state of zero spin. Thus, the literal acceptance of these computations leads to the conclusion that the hypothetical fermion-type mass-4 system is ferromagnetic at absolute zero. However this conclusion is uncertain because (i) contribu-

tions to the energy from 4 and 5 particle cluster integrals may conceivably reverse the sign of $V'''(\rho, 0)$, (ii) the increased range of k values associated with $x = \pm 1$ as compared with $x = 0$ reduces the over-all accuracy of the quadratic approximation for $S(k)$ [Eq. (B1)], (iii) the convolution approximation for $p(1, 2, 3)$ used in evaluating three particle cluster integrals may be inadequate, and (iv) the energy may be lowered at $x = 0$ by pairing effects as in the theory of superconductivity.

Clearly, exchange effects are of crucial importance in the calculation of magnetic properties near absolute zero. It is also clear that an adequate calculation of magnetic properties requires great care and thoroughness in the evaluation of the exchange contribution.

The identity of Eqs. (64) and (71) is easily verified using $e(k)$ given by Eq. (B5). To compute $V'''(\rho, 0)$ of Eq. (68) observe first that Eqs. (66) and (70) yield immediately

$$\begin{aligned}\frac{2}{N}V(\rho, x) &= \frac{3}{5}e_F(\rho)(1+x)^{2/3} \left\{ (1+x) - \left[\frac{1}{2} - \frac{6}{7}B(1+x)^{2/3} \right] \right. \\ &\quad \times (1+x)^2 - \left[\frac{3}{7}B(1+x)^{2/3} - \frac{352}{315}B^2(1+x)^{4/3} \right. \\ &\quad \left. \left. + \frac{472}{735}B^3(1+x)^2 \right] (1+x)^3 \right\}.\end{aligned}\quad (74)$$

Consequently,

$$\begin{aligned}\frac{2}{N}V'''(\rho, 0) &= \frac{3}{5}e_F(\rho) \left[\frac{5}{3} \times \frac{2}{3} - \left(\frac{8}{3} \times \frac{5}{3} \times \frac{1}{2} - \frac{10}{3} \times \frac{7}{3} \times \frac{6}{7} \right) \right. \\ &\quad \left. - \left(\frac{13}{3} \times \frac{10}{3} \times \frac{3}{7} - \frac{352}{315}B^2 \right. \right. \\ &\quad \left. \left. + \frac{17}{3} \times \frac{14}{3} \times \frac{472}{735}B^3 \right) \right].\end{aligned}\quad (75)$$

Equations (68) and (75) reproduce the result of Eq. (72). Thus, the identity of the alternative formulas for effective mass and susceptibility [at least for the assumed form of $S(k)$] is verified.

Equation (44) gives the radial distribution function of the fermion system. A numerical integration is carried out for the mass 4 system with the resulting g_F shown in Fig. 1. The curves are consistent with the expected effect of the exclusion principle in reducing the tendency of particles to cluster near a given particle.

Finally, we repeat that the quasi-particle formalism is needed in this paper only to secure an estimate of the effective mass (or level density) at the Fermi surface. All other quantities relate to properties of the ground-state basis function.

VI. LOW DENSITY HARD SPHERE SYSTEM

In this section, the method developed in previous sections is applied to a system of hard-sphere fermions

¹⁴ J. de Boer, *Physica* **14**, 139 (1948); in *Progress in Low-Temperature Physics*, edited by J. C. Gorter (North-Holland Publishing Company, Amsterdam, 1957), Vol. II, Chap. 1.

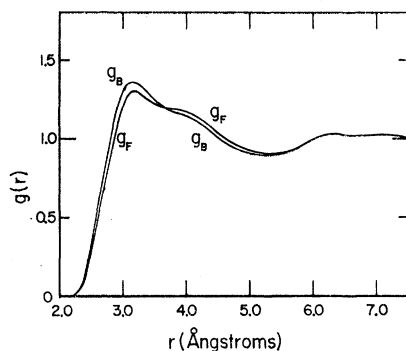


FIG. 1. Radial distribution functions of the Bose and Fermi mass-4 systems at the density of the He II. g_B , the experimental curve; g_F , calculated from Eq. (44), $N_{\pm} = \frac{1}{2}N$.

at low density. The essential function needed in our calculation is the liquid structure factor $S(k)$ of the corresponding Bose system. For such a system Lee, Huang, and Yang¹⁵ find

$$S(k) = k(k^2 + 16\pi a\rho)^{-1/2} + O(k_F^{3/2}a^{3/2}), \quad ka \ll 1, \quad (76)$$

where a is the hard-sphere diameter. For definiteness we consider fermions with twofold spin degeneracy¹⁶ ($k_F^3 = 3\pi^2\rho$). The energy per particle and the velocity of first sound of the corresponding Bose system are also given¹⁵ as

$$\begin{aligned} \frac{1}{N}E_0^B &= 4\pi a\rho \frac{\hbar^2}{2M} \left[1 + \frac{128}{15} \left(\frac{\rho a^3}{\pi} \right)^{1/2} + \dots \right], \\ c_B^2 &= 16\pi a\rho \left(\frac{\hbar}{2M} \right)^2 \left[1 + 32 \left(\frac{\rho a^3}{\pi} \right)^{1/2} + \dots \right]. \end{aligned} \quad (77)$$

Both Eqs. (76) and (77) are subject to the restriction

$$k_F a \ll 1. \quad (78)$$

A meaningful connection between the fermion and boson solutions is possible only if the liquid structure factor $S(k)$ approaches zero at small k in all orders of approximation. The need for this condition has been recognized and emphasized in recent studies of liquid He⁴ using a cluster development procedure.¹⁷ Since the known terms of order $(k_F a)^{3/2}$ in $S(k)$ fail to vanish at $k=0$ we have not attempted to include them in the present calculation. Consistency then requires omission of all three particle cluster integrals in the fermion theory. Thus, we use

$$\begin{aligned} (N/e_F)f_{00}(k_F, y) \\ = (1-y)\{[1 + 8k_F a/3\pi(1-y)]^{-1/2} - 1\} \end{aligned} \quad (79)$$

¹⁵ T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

¹⁶ The discussions can be extended to fermions of arbitrary spin degeneracy. The resulting energy formula agrees with that of reference 18.

¹⁷ Clayton Williams, Ph.D. thesis, Washington University, 1961 (unpublished).

and obtain, retaining only the leading terms in $k_F a$ consistent with the above prescription, the following results from Eqs. (30), (59), (60), and (62):

$$\frac{1}{N}E_0^F = e_F \left[\frac{3}{5} + \frac{2}{3\pi} k_F a + O(k_F^2 a^2) \right], \quad (80)$$

$$M/M^* = 1 + O(k_F^2 a^2 \ln k_F a), \quad (81)$$

$$c_F^2 = \frac{1}{3} \left(\frac{\hbar k_F}{M} \right)^2 \left[1 + \frac{2}{\pi} k_F a + O(k_F^2 a^2) \right], \quad (82)$$

$$\chi_0/\chi = 1 - \frac{2}{\pi} k_F a + O(k_F^2 a^2). \quad (83)$$

The first three formulas as far as they go are in agreement with those given by Lee and Yang¹⁸ and by Abrikosov and Khalalinkov.¹¹ Also, Eq. (83) gives an explicit formula for the susceptibility.

An improved formula for $S(k)$ would permit evaluation of higher order terms in Eqs. (80)–(83) and make possible a more complete comparison with the expansions in reference 11. Exact agreement cannot be expected because unavoidable errors are introduced by use of the Kirkwood approximation (or the alternative convolution approximation) in evaluating the three-particle cluster integrals. Also the restriction of the model function to a single Slater determinant may introduce errors in evaluating terms of higher order in $k_F a$. However, this restriction is not essential to the method. The possibility of a general linear combination^{3,13} of determinants should be explored.

APPENDIX A

Comments on the Cluster Formalism

(a) The linear dependence of $\ln I_N(\beta)$ on N requires, in general,¹ that

$$X_{12\dots n}^{(l)} \sim O(1/N^{n-l-1}). \quad (A1)$$

The index l denotes an additive component in $X_{12\dots n}$ which vanishes unless l relations of linear dependence hold among the wave vectors $\mathbf{k}_1, \mathbf{k}_2, \dots, \mathbf{k}_n$. To obtain estimates of the cluster integrals we introduce the generalized superposition approximation¹⁸

$$p(1, 2, \dots, n) \cong \rho^{n-1} \prod_{1 \leq i < j \leq n} g(r_{ij}), \quad (A2)$$

and find without difficulty that condition (A1) is verified for $n \leq 5$. The first failure occurs at $n=6$ in the result $X_{123456}^{(0)} \sim O(1/N^4)$. We interpret this result as evidence for the increasing inadequacy of the superposition form as n increases, an inadequacy which has catastrophic consequences in this application at $n=6$. Nevertheless, the superposition form, when nothing better is available, can be used to obtain a reasonable and possibly moder-

¹⁸ T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957).

ately accurate evaluation of the four- and five-index cluster integrals.

(b) The derivation of the cluster expansion for $\ln I_N$ will now be examined. We first attempted to extend the method of Iwamoto and Yamada¹ beyond the three-index approximation of Eqs. (21) and (22). Our efforts revealed an inconsistency in the basic equation [reference 1, Eqs. (23) and (25)]. We are indebted to Dr. Iwamoto for an illuminating comment on this point. These equations can be modified, as indicated by Iwamoto, to give consistent results, but we prefer to describe a simple alternative procedure. The discussion may start from the general cluster sum

$$K(1,2,\dots,N) = \sum_{\substack{m_1 \text{ factors} \\ m_2 \text{ factors}}} \{ \dots X_{i_1} \dots X_{j_1} \dots \} \times \{ \dots X_{k_1} \dots X_{m_1} \dots \} \dots, \quad (\text{A3})$$

in which the summation extends over all possible products subject to the conditions (i) no repeated indices; (ii) permutations within a bracket not distinguished; (iii) each X a symmetrical function of its indices, and (iv) $\sum_{l=1}^{\infty} m_l = N$. Let

$$x_{ij\dots hl} = X_{ij\dots hl} / X_i X_j \dots X_h X_l, \quad (\text{A4})$$

$$K(1,2,\dots,N) = X_1 \dots X_N I(1,2,\dots,N);$$

then

$$\begin{aligned} I(1,2,\dots,N) &= 1 + \sum_{m<n} x_{mn} + \sum_{l<m<n} x_{lmn} \\ &+ \sum_{h<l<m<n} (x_{hlmn} + x_{hl}x_{mn} + x_{hm}x_{ln} + x_{hn}x_{lm}) \\ &+ \sum_{h<l<m<n<p} (x_{hlmnp} + x_{hl}x_{mnp} + \dots + x_{np}x_{hlm}) \\ &+ \dots \quad (\text{A5}) \end{aligned}$$

Next, consider the functions generated by omitting all terms in Eq. (A5) containing indices in the set $(ij\dots k)$. The reduced I functions defined by this process are written

$$I_{(ij\dots k)} = I(1, 2, \dots, i-1, i+1, \dots, j-1, j+1, \dots, k-1, k+1, \dots, N). \quad (\text{A6})$$

To simplify the following analysis let $I = I(1,2,\dots,N)$. The reduced function $I_{(q)}$ obtained by omitting a single index q is easily seen to satisfy the equation

$$\begin{aligned} I_{(q)} &= I - \sum_p x_{qp} I_{(qp)} - \sum_{n<p} x_{qp} I_{(qpn)} - \dots \\ &- \sum_{h<l<\dots<p} x_{qp\dots lh} I_{(qp\dots lh)} - \dots - x_{12\dots N}. \quad (\text{A7}) \end{aligned}$$

The substitutions

$$\begin{aligned} I &= e^G, \\ I_{(q)} &= e^{G-\delta G_q}, \\ I_{(qp)} &= e^{G-\delta G_q-\delta G_p}, \text{ etc.}, \end{aligned} \quad (\text{A8})$$

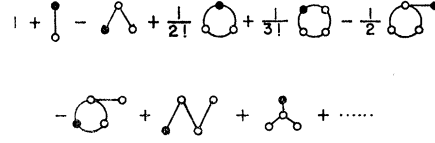


FIG. 2. Diagrammatic representation of Eq. (A11).

transform Eq. (A7) into

$$e^{-\delta G_q} = 1 - \sum_p x_{qp} e^{-\delta G_q - \delta G_p} - \sum_{n<p} x_{qp} e^{-\delta G_q - \delta G_p - \delta G_n} - \dots, \quad (\text{A9})$$

or the equivalent form

$$e^{\delta G_q} = 1 + \sum_p x_{qp} e^{-\delta G_p} + \sum_{n<p} x_{qp} e^{-\delta G_p - \delta G_n} + \dots \quad (\text{A10})$$

We pause to note that δG_p is independent of N ; also that the exponent $G - \delta G_q - \delta G_p$ in Eq. (A8) fails to be exact by terms of order $O(1/N)$. Equations (A9) and (A10) can be solved by an obvious iteration procedure. The result is

$$\begin{aligned} e^{\delta G_q} &= 1 + \sum_p x_{qp} - \sum_{np} x_{qp} x_{pn} + \frac{1}{2} \sum_{np} x_{qp} x_{pn} \\ &+ \frac{1}{6} \sum_{mnp} x_{qmp} - \sum_{mnp} (\frac{1}{2} x_{qp} x_{pnm} + x_{qpn} x_{nm}) \\ &+ \sum_{mnp} (x_{qp} x_{pn} x_{nm} + x_{pq} x_{pn} x_{pm}) + \dots \quad (\text{A11}) \end{aligned}$$

A diagrammatic representation (Fig. 2) is found useful at this stage to characterize the different terms. The meaning of the diagrams is obvious as seen from comparison of Eq. (A11) and Fig. 2; i.e., the open circles refer to free dummy indices of a summation, the solid circles to an index with the value q . This is illustrated in Fig. 3.

Equation (A11) yields the equation represented diagrammatically in Fig. 4. Figure 4 is essentially a difference equation for G . The solution can be found by inspection. We get, up to terms involving five indices (not given in Fig. 4), the equation represented in Fig. 5. The first few terms are in agreement with those given in

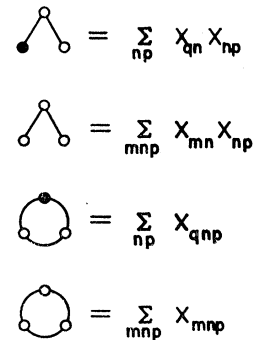


FIG. 3. Meaning of some simple diagrams.

$$\begin{aligned} \delta G_q = & \text{diagram 1} + \frac{1}{2!} \text{diagram 2} - \frac{1}{2} (\text{diagram 3} + 2 \text{diagram 4}) + \frac{1}{3!} \text{diagram 5} \\ & + \frac{1}{2} (2 \text{diagram 6} + 2 \text{diagram 7}) + \frac{1}{3} (\text{diagram 8} + 3 \text{diagram 9}) \\ & - \frac{1}{2} (\text{diagram 10} + \text{diagram 11} + 2 \text{diagram 12}) + \dots \end{aligned}$$

FIG. 4. Diagrammatic equation for δG_q .

reference 1. A discussion of the procedure and the rules for getting the general coefficients in the expansion of G will be given in a forthcoming note.

Observe that four distinct cluster types with four indices appear in the formula for G . The explicit reduction of the cluster integrals using approximate forms for $p(1,2,3)$ and $p(1,2,3,4)$ reveals a large amount of cancellation among the four types, confirming expectations based on the properties of the cluster development for the classical sum of states.¹⁹ Internal cancellation (within a single type) is also important in determining the magnitude of the fourth-order contribution to the physical theory. To illustrate one aspect of the internal cancellation consider the cluster integral X_{hlmn} as defined in Eq. (20). Possibilities for internal cancellation occur in the component of X_{hlmn} generated by the independent pair component of $p(1,2,3,4)$:

$$X_{hlmn}^{(1)} \text{ from } \rho^3 [f(r_{12})f(r_{23}) + f(r_{23})f(r_{31}) + f(r_{31})f(r_{12})]. \quad (\text{A12})$$

The explicit products of two-index X 's cancel out leaving a remainder which vanishes unless relations of linear

$$\begin{aligned} G = & \frac{1}{2!} \text{diagram 1} + \frac{1}{3!} \text{diagram 2} - \frac{1}{2} \text{diagram 3} + \frac{1}{4!} \text{diagram 4} + \frac{1}{2} \text{diagram 5} \\ & + \frac{1}{3} \text{diagram 6} - \frac{1}{2} \text{diagram 7} + \frac{1}{5!} \text{diagram 8} - \frac{1}{2} \text{diagram 9} \\ & - \frac{1}{4} \text{diagram 10} - \text{diagram 11} + \frac{1}{2} \text{diagram 12} + \frac{1}{2} \text{diagram 13} \\ & + \frac{1}{2} \text{diagram 14} - \frac{1}{8} \text{diagram 15} - \frac{1}{6} \text{diagram 16} + \dots \end{aligned}$$

FIG. 5. Diagrammatic equation for G .

dependence such as $\mathbf{k}_{hl} = \mathbf{k}_{mn}$ hold among the four wave vectors [Eq. (A1) and reference 1]. Some internal cancellation occurs within the remainder because of alternations in sign associated with even and odd permutations.

It appears that the ultimate residue remaining in fourth order after all cancellations, internal and external, cannot be estimated without elaborate numerical calculations. We have not attempted these calculations, but venture to suggest that a partial inclusion of fourth-order terms in the physical formulas [as, for example, contributions from $X_{hlmn}^{(1)}$] can hardly be justified before the complete fourth-order contribution has been evaluated.

APPENDIX B

Evaluation of Integrals

The integrals occurring in Eqs. (30), (52), and (57) can be computed by standard methods of numerical integration for given $S(k)$. However, the fact that the experimental $S(k)$ for He^4 is represented fairly well by a quadratic function of k on the range $0 \leq k \leq 2k_F = 1.628 \text{ \AA}^{-1}$ makes possible an analytical evaluation. The formula

$$S(k) = B(k/k_F)^2 = Bx^2, \quad 0 \leq x \leq 2 \quad (\text{B1})$$

with $B = 0.195$ is found to give the integral

$$\int_0^1 S(2k_F x) x^n dx, \quad (\text{B2})$$

with an error of less than 1.3% on the range $1 \leq n \leq 8$. An error of only a few percent can be expected in using Eq. (B1) to evaluate the integrals occurring in Sec. IV. We find, with the aid of Eq. (B1),

$$24 \int_0^1 x^4 u(2k_F x) (1 - \frac{3}{2}x + \frac{1}{2}x^3) dx = -\frac{3}{5} \left(\frac{1}{2} - \frac{6}{7}B \right), \quad (\text{B5})$$

$$\begin{aligned} \left(\frac{3}{8\pi} \right)^3 \int_{x_1, x_2, x_3 \leq 1} x_{12}^2 S(k_F x_{12}) u(k_F x_{13}) u(k_F x_{23}) dx_1 dx_2 dx_3 \\ = -\frac{3}{5} \left(\frac{3}{7}B - \frac{352}{315}B^2 + \frac{472}{735}B^3 \right). \quad (\text{B4}) \end{aligned}$$

Also Eqs. (52), (57), and (58) become

$$\begin{aligned} e(k)/e_F = x^2 + \left[\left(\frac{3}{14} + x^2 + \frac{1}{2}x^4 \right) B - \left(\frac{3}{10} + \frac{1}{2}x^2 \right) \right] - \left[\left(\frac{51}{140} + \frac{1}{2}x^2 + \frac{1}{4}x^4 \right) B \right. \\ \left. - \left(\frac{241}{420} + \frac{221}{140}x^2 + \frac{19}{20}x^4 + \frac{1}{4}x^6 \right) B^2 + \left(\frac{109}{490} + \frac{149}{210}x^2 + \frac{333}{350}x^4 + \frac{3}{10}x^6 \right) B^3 \right], \quad (\text{B5}) \end{aligned}$$

¹⁹ T. L. Hill, *An Introduction to Statistical Mechanics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1960), Chap. 5.

$$\begin{aligned}
\frac{N}{e_F} f_0(\mathbf{k}_1, \mathbf{k}_2) = & \frac{1}{2} x_{12}^2 (B x_{12}^2 - 1) - \frac{1}{4} \left\{ B x_{12}^4 \left[1 - \left(x_1^2 + x_2^2 + \frac{6}{5} \right) B + \left(\frac{3}{5} x_1^2 + \frac{3}{5} x_2^2 + x_1^2 x_2^2 + \frac{4}{5} \mathbf{x}_1 \cdot \mathbf{x}_2 + \frac{3}{7} \right) B^2 \right] \right. \\
& + B (B x_{12}^2 - 1) \left[- \left(\frac{6}{7} + 2x_1^2 + 2x_2^2 + x_1^4 + x_2^4 \right) + B \left(\frac{2}{3} + \frac{16}{7} \mathbf{x}_1 \cdot \mathbf{x}_2 + (x_1^2 + x_2^2) \right. \right. \\
& \left. \left. \times \left(\frac{13}{7} + \frac{8}{5} \mathbf{x}_1 \cdot \mathbf{x}_2 + x_1^2 x_2^2 \right) + 4x_1^2 x_2^2 + \frac{3}{5} (x_1^4 + x_2^4) \right) \right] \right\}, \quad (\text{B6})
\end{aligned}$$

$$\begin{aligned}
\frac{N}{e_F} f_{00}(k_F, y) = & [(2B - 1) + (1 - 4B)y + 2By^2] + \left[\left(-\frac{19}{7}B + \frac{200}{21}B^2 - \frac{884}{105}B^3 \right) \right. \\
& \left. + \left(2B - \frac{296}{35}B^2 + \frac{788}{105}B^3 \right)y + \left(-B + \frac{16}{5}B^2 + \frac{12}{7}B^3 \right)y^2 - \frac{4}{5}B^3y^3 \right]. \quad (\text{B7})
\end{aligned}$$

The integrals needed in Eqs. (59)–(62) are now evaluated:

$$\begin{aligned}
\frac{3N}{4e_F} \int_{-1}^1 f_{00}(k_F, y) dy = & (4B - \frac{3}{2}) - \frac{8}{7} \left(4B - \frac{139}{10}B^2 + \frac{103}{10}B^3 \right), \\
\frac{3N}{4e_F} \int_{-1}^1 f_{00}(k_F, y) y dy = & (\frac{1}{2} - 2B) - \left(B - \frac{148}{35}B^2 + \frac{1844}{525}B^3 \right). \quad (\text{B8})
\end{aligned}$$

Here, in each formula the contributions from the two- and three-particle cluster integrals are grouped separately.