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Wave Propagation along a Magnetic Field in a Warm Plasma

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The steady-state properties of circularly polarized electromagnetic waves propagating along a static magnetic field in a warm uniform plasma are considered. The coupled Maxwell-Boltzmann equations (with ion dynamics neglected) are solved in the presence of short-range collisions. Expressions for the reflection and penetration of the waves into a semi-infinite plasma are obtained. These expressions are explicitly evaluated and discussed for a wide range of physical parameters. The temperature effects are large only (1) deep within the plasma and (2) near electron cyclotron resonance. The effect of temperature is to decrease reflection at frequencies just above the cyclotron frequency and to increase it for frequencies just below the cyclotron frequency. These effects arise from a resonance damping additional to collisional damping and from an upward shift in the effective cyclotron frequency of the hot electron. For frequencies just above the cyclotron frequency, the Poynting's vector in the plasma does not decrease monotonically with distance. The physical origin of this anomalous behavior is discussed.

I. INTRODUCTION

During the past decade there has been considerable interest in the propagation characteristics of electromagnetic waves in a uniform one component plasma immersed in a static magnetic field. Åstrom[1] has analyzed the problem for a cold plasma; that is, he neglected the effects of electron thermal motion. For each direction of propagation of the wave relative to the magnetic field, he found two normal modes. In particular, the normal modes for propagation along the static magnetic field are the right- and left-handed circularly polarized waves whose dispersion relations are given by

$$\epsilon_r \equiv \kappa_r^2/k_0^2 = 1 - [\omega_p^2/\omega(\omega-\omega_b+i\nu_c)], \tag{1}$$

$$\epsilon_l \equiv \kappa_l^2/k_0^2 = 1 - [\omega_p^2/\omega(\omega+\omega_b+i\nu_c)], \tag{2}$$

where $k_0=\omega/c$ is the propagation constant in free space; $\kappa_{r,l}$ is the propagation constant for the right- or left-circularly polarized waves in a cold plasma (we shall denote by $\mathbf{k}$ the propagation vector in a hot plasma); $\omega_b = eB_0/m$ is the electron cyclotron frequency; $\omega_p = (ne^2/m\epsilon_0)^{1/2}$ is the plasma frequency; ν_c is a phenomenological electron collision frequency for momentum transfer; and $\epsilon_{r,l}$ is the effective dielectric constant of the plasma for the two modes of propagation.

Sitenko and Stepanov,[2] Bernstein,[3] Drummond,[4] Pradham,[5] Gershman,[6,7] and Stepanov[8] all analyzed the initial value problem for an infinite uniform plasma taking into account thermal motion but neglecting the effect of collisions, that is, they searched for roots of the dispersion relation. Their analyses are based on the solution of a linearized, collisionless Boltzmann equation with a self-consistent Maxwell field (the Vlasov equation). They found for typical warm plasmas and for frequencies ω far from the cyclotron frequency ω_b, that the dispersion relations (1) and (2) were largely unmodified. Near cyclotron resonance, however, there were significant deviations. More specifically, Pradham found (neglecting collisions) that ϵ_r was complex, that the real part of ϵ_r passed through unity at $\omega=\omega_b$ and that the imaginary part of ϵ_r exhibited a maximum at $\omega=\omega_b$.

The imaginary part of ϵ_r is due to a damping mechanism similar to the Landau damping mechanism characteristic of longitudinal plasma oscillations. Energy is extracted from the wave by a resonant mechanism. Electrons which travel at the modified phase velocity $(\omega-\omega_b)/k$ experience an effective static electric field. As a result, they are accelerated in their orbit and spiral out (or in) around the magnetic lines of force. There is a net energy transfer to the electrons and this results in a damping of the wave. However, in steady state and in the absence of collisions, this damping must be negligibly small since the number of electrons whose velocity is exactly the phase velocity of the wave in question is zero. Electrons whose velocities are near, but not exactly at, the phase velocity will, on the average, gain no net energy from the field unless collisions are present to randomize their ordered motion. Also, as we shall show, in the absence of collisions and in the steady state the linearized theory breaks down for electrons exactly at the phase velocity.

In this paper we will be interested in the *steady-state* properties of transverse electromagnetic waves propagating in a warm plasma along a static magnetic field. Shafranov[9] has analyzed some general aspects of this problem neglecting collisions. The neglect of collisions in the steady state involves, as we have pointed out, certain inconsistencies.[10]

In Sec. II of this paper we consider the steady state,

[1] E. Åstrom, Arkiv Fysik 2, 443 (1950).
[2] A. G. Sitenko and K. N. Stepanov, J. Exptl. Theoret. Phys. (U.S.S.R.) 31, 642 (1956).
[3] I. B. Bernstein, Phys. Rev. 109, 10 (1958).
[4] J. E. Drummond, Phys. Rev. 112, 1461 (1958).
[5] T. Pradham, Phys. Rev. 107, 1222 (1957).
[6] B. N. Gershman, J. Exptl. Theoret. Phys. (U.S.S.R.) 37, 695 (1959).
[7] B. N. Gershman, J. Exptl. Theoret. Phys. (U.S.S.R.) 38, 912 (1960).
[8] K. N. Stepanov, J. Exptl. Theoret. Phys. (U.S.S.R.) 36, 1457 (1959).
[9] B. D. Shafranov, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 1475 (1958).
[10] P. M. Platzman and S. J. Buchsbaum, Phys. Fluids 4, 1288 (1961).

linearized solution of the coupled Maxwell-Boltzmann equations. A discussion of the limits of validity of this solution is presented in Appendix A. The penetration and reflection of a sinusoidal field into a semi-infinite plasma is computed. In Sec. III we explicitly evaluate these expressions for a wide range of physical plasma parameters.

II. SOLUTION OF THE COUPLED MAXWELL-BOLTZMANN EQUATIONS

We consider a one-component plasma filling uniformly the semi-infinite half space $z>0$. The plasma is immersed in a uniform static magnetic field $\mathbf{B}_0$ which is oriented in the positive z direction. We are interested in computing the reflection and penetration properties of a transverse, circularly polarized wave incident normally upon the plasma from the left. Without loss of generality we will assume that the wave has a monochromatic time dependence and is of the form $\mathbf{E}=\mathbf{E}(z)e^{-i\omega t}$.

In general, as a result of the thermal motion of the electrons, the plasma conductivity is a nonlocal integral operator. That is to say, a field with a β component at at point $\mathbf{r}'$ in the plasma will produce a current with an α component at another point $\mathbf{r}$ in the plasma, with magnitude $\sigma_{\alpha\beta}[\mathbf{r},\mathbf{r}',\omega]E_\beta(\mathbf{r}')$.

$$J_\alpha(\mathbf{r},\omega)=\int\sigma_{\alpha\beta}[\mathbf{r},\mathbf{r}',\omega]E_\beta(\mathbf{r}')d\mathbf{r}'. \tag{3}$$

In an infinite medium $\sigma_{\alpha\beta}[\mathbf{r},\mathbf{r}',\omega]$ is evidently only a function of $|\mathbf{r}-\mathbf{r}'|$. In the presence of boundaries $\sigma_{\alpha\beta}[\mathbf{r},\mathbf{r}',\omega]$ may be a function of $\mathbf{r}-\mathbf{r}'$ and of $\mathbf{r}'$. In our problem, where $\mathbf{B}_0$ is perpendicular to the plane boundary, the conductivity is clearly a function only of the coordinate difference in the (x',y') plane. It may depend on z', the distance from the boundary, if the boundary affects the velocity distribution in the z' direction. If the electrons are specularly reflected at the boundary ($z=0$), the velocity distribution is unaffected by the boundary since (1) the magnetic field does not affect the z component of electron velocity, and (2) the electrons for $z<0$ in the infinite medium are replaced by their equivalent images in the semi-infinite medium problem.[11] If the magnetic field had a component in the (x,y) plane, or, if the electrons were diffusely reflected from the boundary, these arguments would not hold. The solution of Maxwell equations is then a much more difficult task as Fourier transform methods no longer simplify the problem.

Hereafter we assume that $\sigma_{\alpha\beta}$ is only a function of $|\mathbf{r}-\mathbf{r}'|$. Then, in Fourier space Eq. (3) becomes an algebraic relation,

$$J_\alpha(\mathbf{k},\omega)=\sigma_{\alpha\beta}(\mathbf{k},\omega)E_\beta(\mathbf{k},\omega), \tag{4}$$

[11] G. H. Reuter and E. H. Sondheimer, Proc. Roy. Soc., (London) **A195**, 336 (1949).

where $\sigma_{\alpha\beta}(\mathbf{k},\omega)$ is the plasma conductivity tensor as computed for the infinite medium. A determination of $\sigma_{\alpha\beta}(\mathbf{k},\omega)$ thus permits a complete solution of the steady-state boundary value problem, provided specular reflection of electrons at the boundary is assumed.

In order to determine $\sigma_{\alpha\beta}(\mathbf{k},\omega)$ we use the Boltzmann equation,

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f-\frac{e}{m}[\mathbf{E}+\mathbf{v}\times\mathbf{B}]\cdot\frac{\partial f}{\partial\mathbf{v}}=\nu_c(f_0-f). \tag{5}$$

The proper collision integral has been approximated by a phenomenological velocity independent collision frequency, ν_c, which gives the relaxation of f to a uniform equilibrium distribution f_0. More properly, the right-hand side of (5), if it is to conserve particles, should relax to a local density rather than to a uniform density.[10] Here, we are concerned only with transverse waves where there are no local density fluctuations.

Equation (5) usually is linearized by setting $f=f_0+f_1$, assuming that f_1 is proportional to E, and neglecting terms of the order E^2. In the steady-state problem the validity of this assumption must be carefully examined for that class of electrons which is in resonance with the wave, i.e., whose z component of velocity is near the modified phase velocity of the wave $(\omega-\omega_b)/k$. This question is considered in detail in the Appendix. We show there that in the absence of collisions there exist electrons for which the linearization procedure is never valid no matter how small E is. However, in presence of collisions the linearization procedure is valid for all electrons provided

$$\nu_c>eE/m\langle v^2\rangle^{1/2}, \tag{6}$$

where $\langle v^2\rangle$ is some proper average of the square of the thermal velocity. The above inequality assures that the effective static electric field which the resonant electrons experience, acts for times short compared with the time it takes to heat those electrons. We will henceforth assume that inequality (6) is satisfied and analyze the problem on the basis of a linearized theory.

For right- or left-handed circularly polarized transverse wave, f_1 is given by

$$f_1=(e\mathbf{E}/m)_{r,l}\cdot(\partial f_0/\partial\mathbf{v})/[\nu_c-i(\omega+\omega_b-\mathbf{v}\cdot\mathbf{k})].$$

The conduction current $\mathbf{J}=\int\mathbf{v}f_1d^3v$ is then likewise left- or right-handed circularly polarized. It is convenient to define the corresponding scalar conductivities $\sigma_{r,l}$ by $J_{r,l}=\sigma_{r,l}E_{r,l}$. These conductivities are

$$\sigma_{r,l}=\frac{i\omega_p^2}{k}\int\frac{f_0d^3v}{v_z-u_{r,l}}, \tag{7}$$

where

$$u_{r,l}=(\omega\mp\omega_b+i\nu_c)/k, \tag{8}$$

and k is the component of $\mathbf{k}$ in the direction of propagation of the wave. The quantity f_0 is the unperturbed

electron distribution function and it is taken to be normalized to unity.

The solution of the boundary value problem for the semi-infinite half-space is equivalent to finding the field in an infinite medium excited by surface currents and charges at the surface, $z=0$.[12] The Fourier component of the electric field in the infinite medium is in turn determined from the wave equation

$$\mathbf{k}\times\mathbf{k}\times\mathbf{E}(\mathbf{k},\omega)-\omega^2\mu_0\mathbf{D}(\mathbf{k},\omega)=(i\omega\mu_0)\mathbf{J}^s(\mathbf{k},\omega), \quad (9)$$

where

$$D_\alpha(\mathbf{k})=\epsilon_{\alpha\beta}(\mathbf{k},\omega)E_\beta(\mathbf{k},\omega), \quad (10)$$

and

$$\epsilon_{\alpha\beta}(\mathbf{k},\omega)=\delta_{\alpha\beta}+i\sigma_{\alpha\beta}(\mathbf{k},\omega)/(\omega\epsilon_0). \quad (11)$$

The current $\mathbf{J}^s(\omega,\mathbf{k})$ is the exciting current in the plane $z=0$ and is related to the value of the transverse rf magnetic field in this plane. It is a fictitious current and is introduced purely for mathematical convenience. Thus, for transverse circularly polarized waves

$$\left(k^2-\frac{\omega^2}{c^2}\epsilon_{r,l}\right)E_{r,l}(\mathbf{k},\omega)=+\frac{i}{\pi}\left(\frac{\omega}{c}\right)\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}H_{r,l}{}^0(\omega)\delta(k_\perp), \quad (12)$$

where $H_{r,l}{}^0$ is the value of the rf magnetic field at the boundary and $k_\perp$ is the component of $\mathbf{k}$ in the plane perpendicular to $\mathbf{B}_0$.[13]

For the transverse waves it is clear that $\epsilon(\mathbf{k},\omega)$ is only a function of ω and of k, the component of $\mathbf{k}$ along the direction of propagation [Eq. (7)]. Since $\mathbf{E}(\mathbf{r})=\int \exp(i\mathbf{k}\cdot\mathbf{r})\mathbf{E}(\mathbf{k})d\mathbf{k}$, the integral over the two components of $\mathbf{k}$ perpendicular to the z direction may be performed. The field inside the medium is a function of z only, and is given by

$$E_{r,l}(z)=+\frac{i}{\pi}\left(\frac{\mu_0}{\epsilon_0}\right)^{1/2}\int_{-\infty}^{+\infty}H_{r,l}{}^0\frac{(\omega/c)e^{ikz}dk}{[k^2-\omega^2\epsilon_{r,l}(k,\omega)/c^2]}. \quad (13)$$

The path of integration is along the real k axis. We choose units measuring length in terms of k_0^{-1} and take ϵ_0 and μ_0 equal to unity. In these units the impedance of the plasma at $z=0$ is,

$$Z_{r,l}=\frac{E_{r,l}}{H_{r,l}{}^0}=+\frac{i}{\pi}\int_{-\infty}^{+\infty}\frac{dk}{[k^2-\epsilon_{r,l}(k,\omega)]}. \quad (14)$$

When ϵ is independent of k as in a cold plasma [$f_0(v)=\delta^3(v)$], then

$$E_{r,l}(z)=\epsilon_{r,l}{}^{-1/2}H_{r,l}{}^0\exp(i\epsilon_{r,l}{}^{1/2}z), \quad (15)$$

and

$$Z_{r,l}=\epsilon_{r,l}{}^{-1/2}. \quad (16)$$

[12] W. Panofsky and M. Phillips, *Classical Electricity and Magnetism* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955).

[13] In position space the exciting surface current is localized on the plane $z=0$ and has the value $2\hat{u}_z\times\mathbf{H}^0\delta(z)$. Since $\delta(z)=(2\pi)^{-1}\int e^{i\mathbf{k}\cdot\mathbf{r}}\delta(k_\perp)d^3k$, Eq. (12) follows directly.

III. EVALUATION AND DISCUSSION OF RESULTS

The left-hand circularly polarized wave exhibits little explicit temperature effects since the "modified" phase velocity of the wave $(\omega+\omega_b)/k$ for the important k's is generally much greater than the mean thermal velocity of the electrons. Thus, we will consider only the right-hand circularly polarized wave, and will henceforth drop the subscript r identifying it. For this wave near cyclotron resonance, the important waves travel sufficiently slowly that a strong interaction with a sufficient number of electrons is possible.

Equations (7) and (11) define an $\epsilon(k,\omega)$ which is not an analytic function of k. The integral

$$I=\int_{-\infty}^{\infty}\frac{f_0d^3v}{[v_z-u]} \quad (17)$$

defines two functions, one analytic in the upper-half u plane, and one analytic in the lower-half u plane. Equation (13) which determines the field E, may be conveniently split into two parts

$$E(z)=\frac{i}{\pi}H^0\left[\int_{-\infty}^{0}\frac{e^{ikz}dk}{[k^2-\epsilon^L(k,\omega)]}+\int_0^{\infty}\frac{e^{ikz}dk}{[k^2-\epsilon^U(k,\omega)]}\right]. \quad (18)$$

Here, $\epsilon^U(k,\omega)$ and $\epsilon^L(k,\omega)$ are the analytic functions obtained from I [Eq. (17)] by assuming u is the upper- or lower-half plane, respectively. As $\epsilon^L(k,\omega)=\epsilon^U(-k,\omega)$,

$$E(z)=\frac{i}{\pi}H^0\left\{\frac{1}{2}\int_{-\infty}^{+\infty}e^{+ikz}\left[\frac{1}{\lambda(k)}+\frac{1}{\lambda(-k)}\right]dk+\int_0^{\infty}\cos kz\left[\frac{1}{\lambda(k)}-\frac{1}{\lambda(-k)}\right]dk\right\}. \quad (19)$$

The quantity, $\lambda(k)=k^2-\epsilon^U(k,\omega)$, is an analytic function over the entire u plane. The first integral in Eq. (19) is the usual sum over normal modes obtained from a solution of the dispersion relation $\lambda(k)=0$ and $\lambda(-k)=0$. The second term, which vanishes in the limit of zero temperature, is a non-normal mode-like term. It generates a field which, in general, does not vary exponentially with distance with the exponent proportional to the distance. We must conclude then that a normal mode analysis for a warm plasma is not strictly valid. In more physical terms this means that it is impossible to excite by any type of incident field a single "mode" which possesses a purely exponential space dependence.

In order to evaluate Eq. (18), some choice must be made for the equilibrium distribution function $f_0(v)$. A Maxwellian distribution is the obvious choice. However, the numerical calculations are considerably simpli-

fied if a one-dimensional polynomial distribution of the form

$$f_0^n(v_z)=N/(v_z^2+\alpha^2)^n \tag{20}$$

is chosen. The variables v_x and v_y are assumed to be integrated out; N is a normalization constant and α is determined by setting $\langle v^2\rangle=(3KT/m)$. We shall justify this choice for $f_0(v_z)$ later.

With Eq. (20) for a distribution function, $\lambda(k)$ becomes

$$\lambda(k)=k^2-1+\frac{\omega_p^2}{\omega^2}\sum_{j=1}^{n}\frac{(i\alpha k)^{j-1}(j-1)!}{[i\alpha k+\Delta]^j(2j-3)!!}, \tag{21}$$

where

$$\alpha=[(2n-3)KT/mc^2]^{1/2}, \tag{22}$$

$$\Delta=(1-\omega_b/\omega+i\nu_c/\omega), \tag{23}$$

and $(2j-3)!!\equiv(2j-3)(2j-5)\cdots5\times3\times1$. Thus, $\lambda(k)$ may be written as a ratio of two polynomials:

$$\lambda(k)=G(k)/F(k). \tag{24}$$

The polynomial function $F(k)$ is two orders lower in k than $G(k)$ so that the ratio F/G may be broken up into a sum of partial fractions with linear denominators:

$$\frac{F(k)}{G(k)}\equiv\sum_{i=1}^{n+2}\frac{a_i}{k-k_i}, \tag{25}$$

where the k_i are the roots of $\lambda(k)=0$.

In terms of the roots k_i and the coefficients a_i, the field and the impedance Z are easily evaluated and are found to be,[14]

$$E(z)=H^0\frac{2i}{\pi}\sum_{j=1}^{n+2}a_j\{[\mathrm{Si}(k_jz)-\tfrac{1}{2}\pi]\sin(k_jz)-\cos(k_jz)\,\mathrm{Ci}(k_jz)\}, \tag{26}$$

$$Z=-\frac{2i}{\pi}\sum_{i=1}^{n+2}a_i\ln(-k_i). \tag{27}$$

The quantities Si and Ci are the sine and cosine integrals of complex argument.

An examination of the asymptotic form $(z\to\infty)$ of Eq. (26) indicates that the nonexponential part of the field falls off with z at least as fast as $(1/z^2)$. The coefficients of the $1/z^2$ terms are, at most, of order $(KT/mc^2)^{1/2}$. This weak field, which extends deep into the plasma, is carried there by electrons traveling at high velocities for times of the order of $1/\nu_c$. The particular form of this field at large z is strongly determined by the choice of the distribution function. Shafranov[9] has shown that for a Maxwell distribution the field at infinity depends on z as $\sim\exp(-z^{2/3})$.

Far from resonance where

$$\delta\equiv|[(\omega-\omega_b+i\nu_c)/k](m/KT)^{1/2}|\gg1, \tag{28}$$

and near the boundary, the field is nearly a pure exponential, the deviations from exponential being of order (KT/mc^2). The single wave vector k differs only slightly from its value in a cold plasma, κ. For $n=2$ in Eq. (20) we find that

$$k=\kappa(1+\epsilon_1), \tag{29}$$

with

$$\epsilon_1=-\frac{\omega_p^2}{2\omega^2}\frac{1}{\Delta^3}\frac{KT}{mc^2}. \tag{30}$$

For $\delta\gg1$ it is possible to make an asymptotic expansion of I [Eq. (17)] for a Gaussian distribution function. Near the boundary the field is again exponential with

$$k=\kappa(1+\epsilon_2), \tag{31}$$

and

$$\epsilon_2=-\frac{\omega_p^2}{2\omega^2}\frac{1}{\Delta^3}\frac{KT}{mc^2}. \tag{32}$$

Since ϵ_2 is identical to ϵ_1, we conclude that far from resonance and near the boundary the Lorentzian distribution function yields the same field dependence as a Gaussian.

Near cyclotron resonance the crucial phase velocities in Eq. (17) are sufficiently small so that again the Lorentzian-like distribution function with the correct $\langle v^2\rangle$ is an excellent substitute for a Gaussian, provided the exact functional form of E at infinity is not required.

Figure 1 is a plot of the reflection coefficient

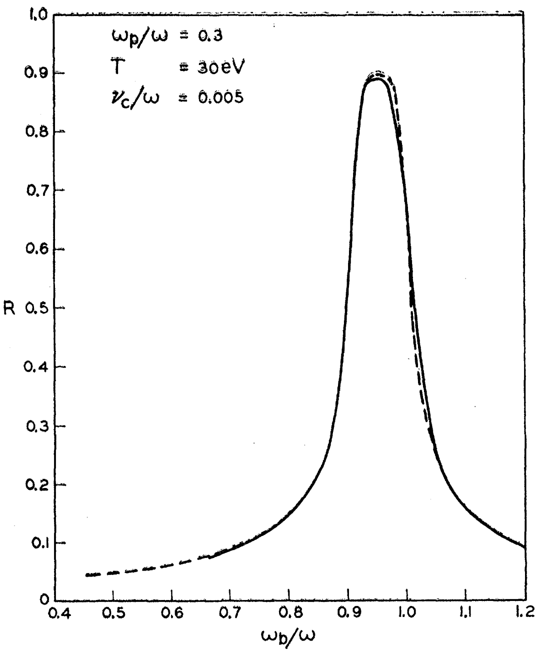

FIG. 1. Plot of the field reflection coefficients $R(0)$ (dashed curve) and $R(T)$ (solid curve) as a function of ω_b/ω for fixed values of ω_p/ω, T, and ν_c/ω.

[14] Bierens de Haan, *Nouvelles Tables D'Integrales Definies* (Leide, 1867), p. 223.

$R \equiv |(1-Z)/(1+Z)|$, for the warm plasma as a function of ω_b/ω for a fixed value of ω_p/ω, KT/mc^2, and ν_c/ω. The dashed curve is the reflection coefficient $R(0)$ for the cold plasma and the solid curve is the reflection coefficient $R(T)$ for the warm plasma. For the finite temperature case a distribution function of polynomial form [Eq. (20)] with $n=2$ was chosen. This is the lowest value of n for which a finite $\langle v^2 \rangle$ exists. The roots of $\lambda(k)$ become the solution of a quartic equation with complex coefficients. These roots were found analytically and were evaluated numerically for a large number of plasma parameters on an IBM 7090. We note first that $R(0)$ is large only for $1-\omega_p^2/\omega^2 \leq \omega_b/\omega \lesssim 1$. In this range of ω_b/ω the plasma is cut off because ϵ_r, as given by Eq. (1), is negative so there is no propagation. We note that a finite temperature T modifies the reflection coefficient only near cyclotron resonance, $\omega_b/\omega \simeq 1$. Of course, this is so because only for $\omega_b/\omega \simeq 1$ is the phase velocity of the wave sufficiently small that the effect of finite random velocity of the electrons is appreciable.

Figures 2, 3, and 4 are plots on an expanded scale of the difference ΔR between the reflection coefficients at zero temperature $R(0)$ and finite temperature $R(T)$, $\Delta R = R(T) = R(0)$. In all cases which we have studied, the effect of the temperature is to decrease the reflectivity for $\omega_b/\omega < 1$ and increase it for $\omega_b/\omega > 1$ in such a manner that a plot of ΔR against ω_b/ω results in a "dispersion like" curve at $\omega_b/\omega \simeq 1$.

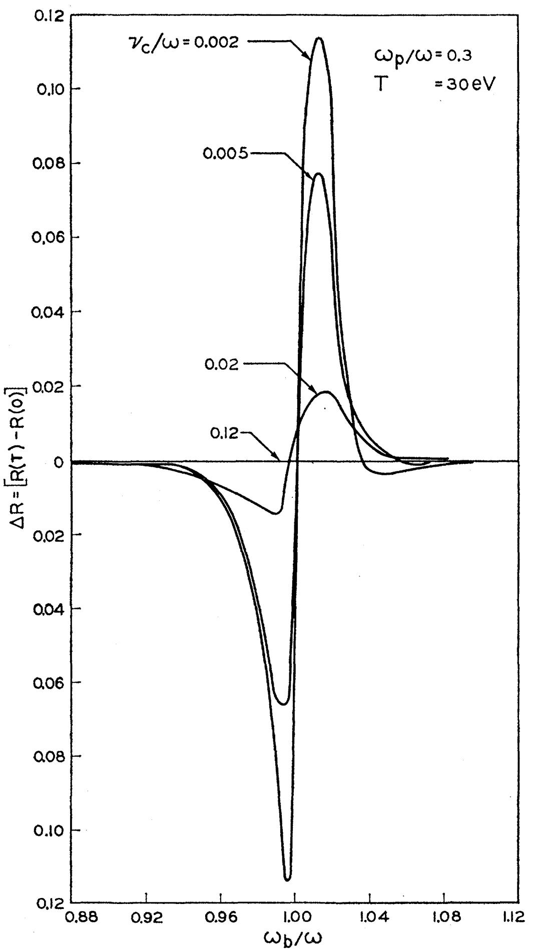

FIG. 2. Plot of the difference of the field reflection coefficients $R(0)$ and $R(T)$ as a function of ω_b/ω at fixed ω_p/ω and T for various values of ν_c/ω.

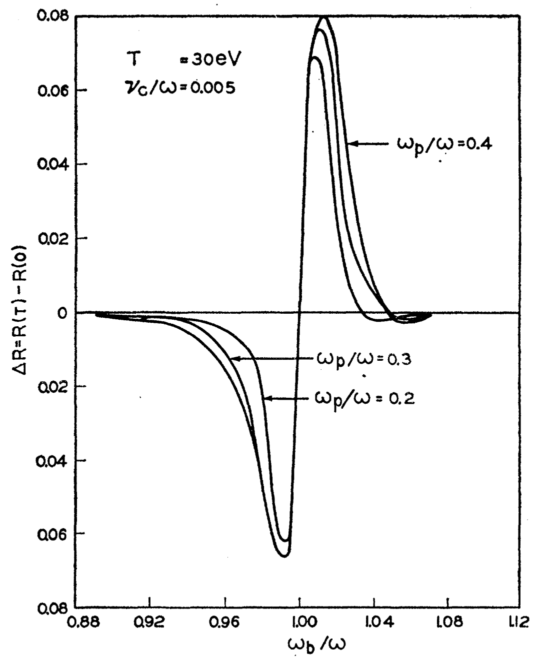

FIG. 3. Plot of the difference of the field reflection coefficients $R(0)$ and $R(T)$ as a function of ω_b/ω at fixed T and ν_c/ω for various values of ω_p/ω.

The sequence of curves starting with Fig. 2 depicts the behavior of this "dispersion" curve as one of the relevant plasma parameters is varied. Figure 2 shows the dependence of ΔR on the collision frequency. As one would expect, ΔR decreases rapidly with increasing ν_c. At large ν_c the pole of the denominator in Eq. (17) is far from the path of integration even for $\omega \simeq \omega_b$. Another way of saying this is that as ν_c increases, the limiting phase velocity near resonance is proportional to ν_c rather than to $(\omega - \omega_b)$ so that the velocity of the important waves are not small. That is, δ of Eq. (28) remains large. A small ν_c enhances the hot-electron effects; the electrons drifting in synchronism with the wave experience the effective static field of the wave for a long period of time. In fact, for sufficiently small ν_c, our linearized description breaks down.

Figure 3 depicts the behavior of ΔR with the plasma frequency. The dependence on ω_p is relatively weak. However, the ΔR curves do spread out somewhat and increase in magnitude with increasing ω_p. This results from a slowing down of the waves with increasing plasma densities.

In Fig. 4 is shown the dependence of the ΔR curve on the temperature. As T is increased, the ΔR curves

increase in magnitude and spread out. There are relatively more high-velocity electrons at the higher temperatures so that there will be larger resonant effects. These effects exist for values of ω_b/ω removed from $\omega_b/\omega=1$ since at higher temepratures the waves can have a larger phase velocity and still resonante with a mean thermal electron.

It is possible to describe the rather complicated dynamics which gives rise to the behavior of the reflection coefficient in terms of two simple macroscopic parameters. We conclude from an examination of the numerical results that the warm plasma, as far as the reflection coefficient is concerned, may be described in terms of parameters of the cold plasma with (a) a modified collision frequency, and (b) a net shift in the cyclotron resonance frequency.[5]

The additional damping mechanism is, of course, the "resonance damping," the analog of the Landau damping phenomenon for longitudinal oscillations. It can be

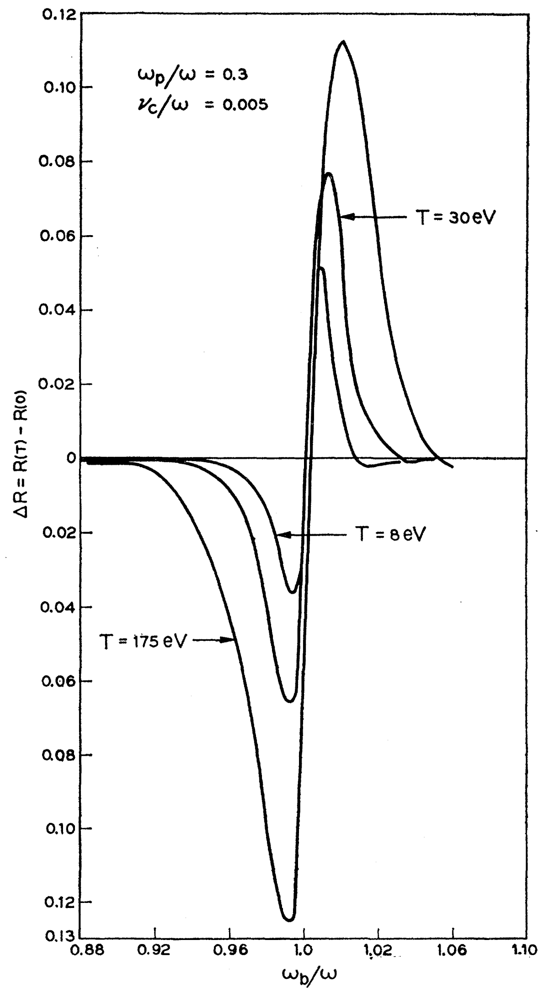

FIG. 4. Plot of the difference of the field reflection coefficients $R(0)$ and $R(T)$ as a function of ω_b/ω at fixed ω_p/ω and ν_c/ω for various values of T.

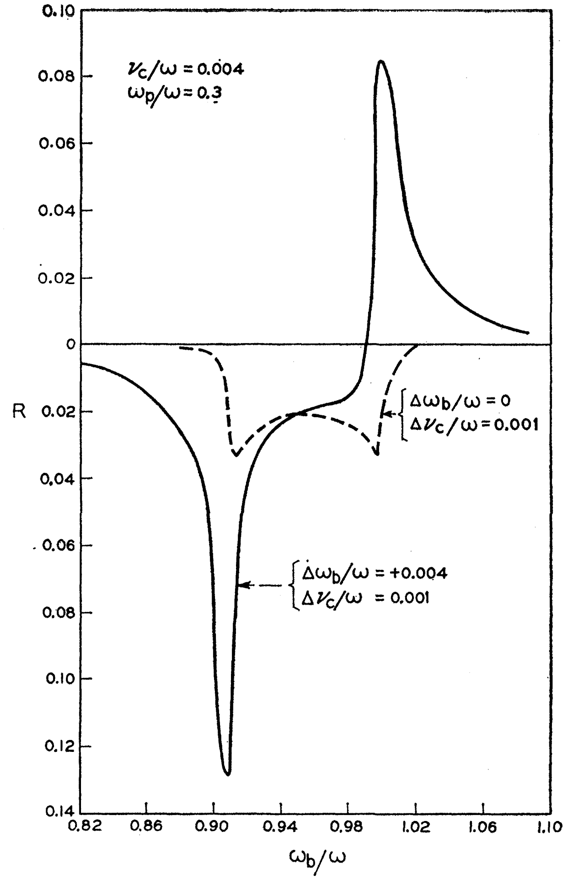

FIG. 5. Plot of the difference of the field reflection coefficients in a cold plasma as a function of ω_b/ω for a shifted collision frequency (dashed curve) and for a shifted collision frequency and cyclotron frequency (solid curve).

crudely represented by ascribing to the electrons an effective collision frequency ν_{eff}, where $\nu_{\text{eff}}=\nu_c+\nu_{\text{res}}$, with ν_{res} being an increasing function of T, ω_p, $1/\nu_c$, and a very strongly increasing function of $1/|\omega-\omega_b|$.

The shift in the cyclotron resonant frequency can be represented by $\omega_b'=\omega_b+\Delta\omega_b$ at resonance, and arises from the fact that only the electrons traveling with the wave are strictly at resonance and they experience a field at the Doppler-shifted frequency. The quantity $\Delta\omega_b$ is primarily a function of the temperature although it depends weakly on the other plasma parameters.

In Fig. 5 we plot the difference of two cold-plasma reflection curves. The dashed curve is the difference in reflection coefficients $R(\omega_b,\nu_c+\nu_{\text{res}})-R(\omega_b,\nu_c)$; the solid curve is $R(\omega_b+\Delta\omega_b,\ \nu_c+\nu_{\text{res}})-R(\omega_b,\nu_c)$. No attempt is made here to fit the actual results in a hot plasma since ν_{res} and $\Delta\omega_b$ were chosen to be independent of the plasma parameters, which is not correct. However, this crude computation indicates that increasing ν_c only is not sufficient to produce the "dispersion like" curves of Figs. 2–4. One must shift the cyclotron fre-

quency by a positive amount to produce the "dispersion-like" behavior (solid curve).

The reflectivity, being an average property of the plasma is quantitatively insensitive to the detailed microscopic processes taking place in a warm plasma. The field itself is a more sensitive probe of such effects. Figure 6 is a plot of the time-average Poynting's vector as a function of distance into the plasma for values of the plasma parameters corresponding to the maximum positive point $\omega_b/\omega=1.01$, on the $T=30$ eV "dispersion" curve of Fig. 4. The Poynting's vector, at least over the range of distances plotted, is approximately exponential and extremely close to its value in a cold plasma. At large distance the field is no longer exponential due to the presence of high-velocity electrons in the distribution function. The dependence at large distances is, as we have pointed out, a critical function of the tail of the equilibrium distribution function. Since our Lorentzian distribution is incorrect at high velocity, we attribute no significance to the behavior of the field at large distances from the boundary.

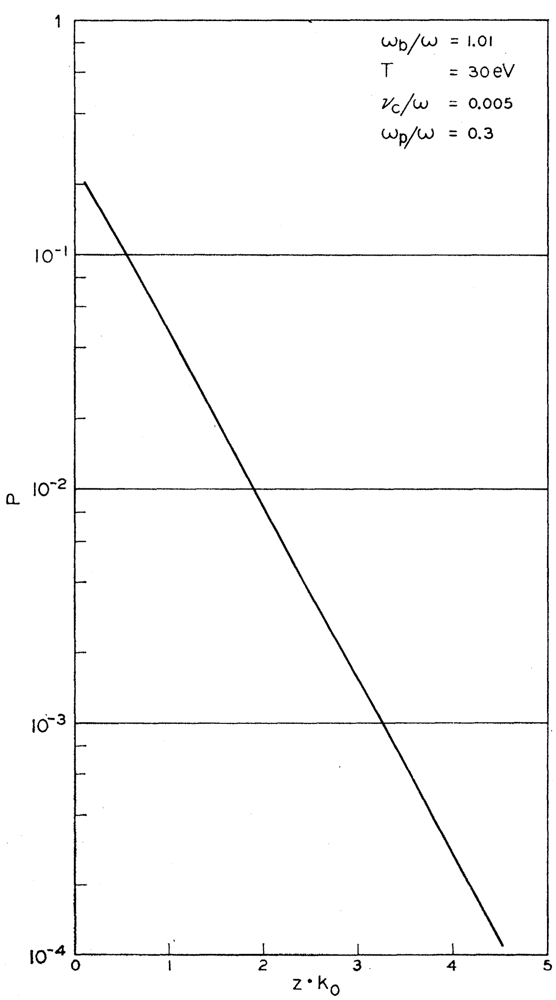

FIG. 6. Plot of the time-averaged Poynting's vector P as a function of distance z in the plasma for fixed ω_b/ω.

Fig. 7 is a plot of the Poynting's vector for the same plasma parameters as in Fig. 6, except that ω_b/ω has been set equal to 0.99, the maximum negative point on the $T=30$ eV dispersion curve of Fig. 4. The behavior here is quite striking. The Poynting's vector falls off rapidly with distance near the boundary, becomes negative, passes through a minimum, returns to a maximum and finally decays uniformly to zero at infinity. In the region near the minimum of the Poynting's vector, the divergence of the Poynting's vector is positive. At zero temperature, where there is a point-to-point relation

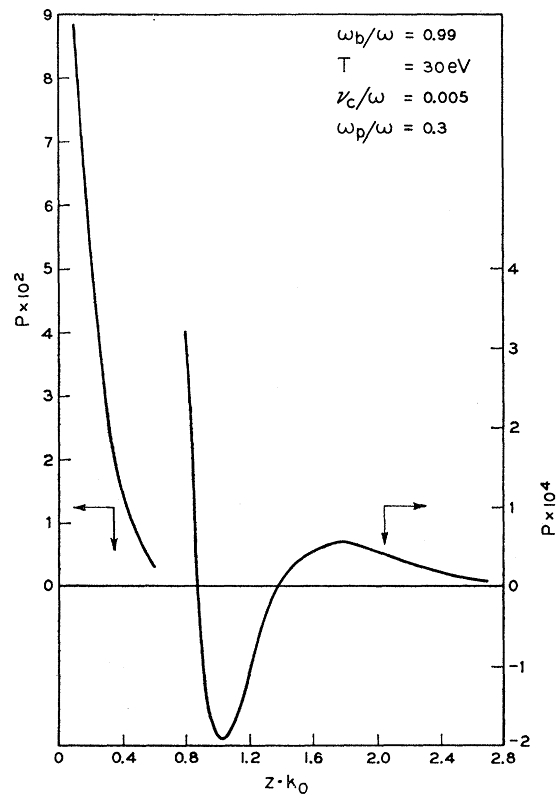

FIG. 7. Plot of the time-averaged Poynting's vector P as a function of distance z in the plasma for fixed ω_b/ω.

between field and current, such behavior of the electromagnetic energy flow is not possible since the divergence of the Poynting's vector is negative definite everywhere. There is a net Joule heating of the plasma in each volume element, forcing the Poynting's vector and the field to decay monotonically with distance.

For a medium in which ϵ depends on k and ω, the energy conservation equation for the time-averaged Poynting's vector $\langle \mathbf{P}_\omega(r)\rangle$ at a fixed frequency ω is

$$\nabla\cdot\langle \mathbf{P}_\omega(\mathbf{r})\rangle = -\tfrac{1}{2}\,\mathrm{Re}\left\{E_\alpha(\mathbf{r})\int \sigma_{\alpha\beta}[\mathbf{r}-\mathbf{r}',\omega]E_\beta(\mathbf{r}')d\mathbf{r}'\right\}, \quad (33)$$

where

$$\langle \mathbf{P}_\omega(\mathbf{r})\rangle = \tfrac{1}{2}\,\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*). \tag{34}$$

In a cold plasma $\sigma[\mathbf{r}-\mathbf{r}',\omega]=\delta(\mathbf{r}-\mathbf{r}')\sigma(\omega)$, where σ is the plasma conductivity. Since Re σ is greater than zero, (if the electron distribution function is monotonically decreasing with velocity) $\langle \mathbf{P}_\omega(r)\rangle$ is negative definite. Mathematically, the general requirements on $\sigma[\mathbf{r}-\mathbf{r}',\omega]$ and $\mathbf{E}(\mathbf{r})$ such that $\langle \mathbf{P}_\omega(\mathbf{r})\rangle$ be less than zero are sufficiently severe that under a given set of circumstances $\langle \mathbf{P}_\omega(\mathbf{r})\rangle$ can, in fact, become positive. It corresponds to the situation where the electromagnetic energy flow in the plasma increases at the expense of the kinetic-energy flow. Such kinetic-energy flows are not considered in this paper.[15]

Physically, the mechanism for the transfer of energy from the plasma to the electromagnetic field is the following: The electric field at a point creates at that point a current with an in-phase component so that there is a net transfer of energy from the field to the electrons. However, if σ is a function of $|\mathbf{r}-\mathbf{r}'|$ the current can drift by virtue of the finite random velocity of the electrons to another point in the plasma where the field is of opposite phase. The current created by a field at one point may then give up energy to the field at another point in the plasma. The net energy transfer to the electromagnetic field at a point is determined by an average over the infinite half-space. Figure 7 indicates that near cyclotron resonance, where the medium is strongly dispersive, this averaging process gives a net negative energy transfer from the wave to the electrons in some region of the plasma. This region begins approximately one plasma wavelength inside the plasma.[16] The field must undergo a phase reversal in order to produce this effect. At the boundary there can be no phase reversal.

For $\omega_b/\omega=1.01$, Fig. 6, this anomalous behavior is not present. The field inside the plasma at this value of ω_b/ω is not decaying as rapidly as it does when $\omega_b/\omega=0.99$. The medium is qualitatively as dispersive at $\omega_b/\omega=1.01$ as at $\omega_b/\omega=0.99$. However, the gradient of the field is important since there must be relatively large fields of opposite phase nearby in space in order that the divergence of the Poynting's vector become positive.

Figure 8 is a plot of the time average of E^2. The mean square E for $\omega_b/\omega=0.99$ exhibits an anomalous behavior in the region where the Poynting vector is negative. For $\omega_b/\omega=1.01$ the field falls off exponentially with distance and is nearly equal to the field in cold plasma.

It is conceivable, although highly improbable, that the anomalous behavior of the Poynting's vector results

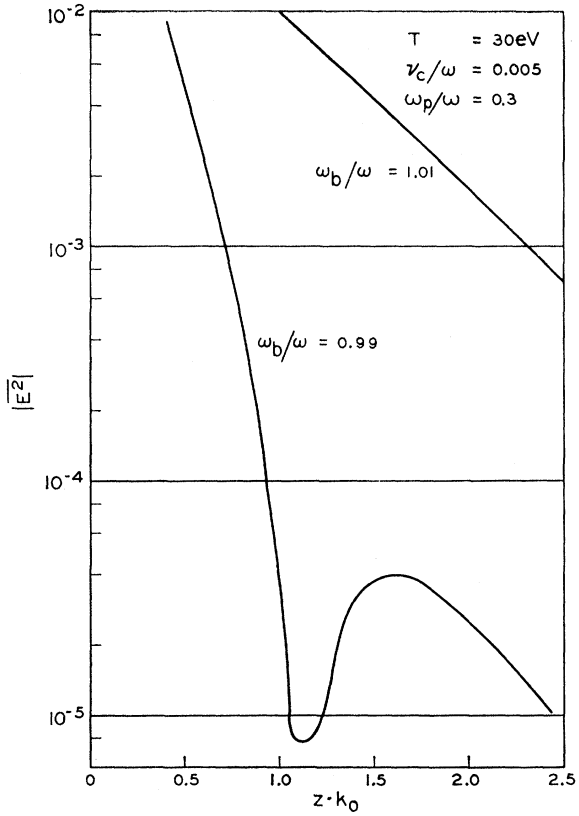

FIG. 8. Plot of the time-averaged square of the electric field E as a function of distance z in the plasma for fixed ω_b/ω.

from the excessive number of high-velocity electrons in the tail of the Lorentz distribution. It is clear from the expression for the conductivity, Eq. (7), that the dominant contribution to the dispersive character of the dielectric constant near cyclotron resonance, arises from electrons near the mean thermal velocity $(KT/m)^{1/2}$ and not from electrons in the tail of the distribution. For these electrons, the Lorentzian distribution is an excellent approximation to a Maxwellian.

ACKNOWLEDGMENTS

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APPENDIX A

In this Appendix we shall discuss the limits of validity of linearization of the Boltzmann equation.

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\nabla f-e[\mathbf{E}+\mathbf{v}\times\mathbf{B}]\cdot\frac{\partial f}{\partial \mathbf{v}}=\nu_c(f_0-f), \tag{A1}$$

where $\mathbf{E}$ is the field of a left- or right-handed circularly polarized wave.

If the dominant collision mechanism is between electrons of mass m and heavy centers of mass M, then the

[15] Conservation theorems associated with energy flow in warm plasmas have recently been developed by A. Bers, *Waves in Anisotropic Plasmas*, edited by W. P. Allis, S. J. Buchsbaum, and A. Bers [Technology Press, Cambridge, Massachusetts (to be published)].

[16] For $\omega_p/\omega=0.3$, $\omega_b/\omega=0.99$, and $\nu_c/\omega=0.005$, the plasma wavelength is approximately a free-space wavelength.

use of a phenomenological velocity independent collision frequency ν_c is accurate to order m/M, when used to compute quantities which depend on the first moment of the distribution function, i.e., the electric current. However, this approximation for the collision integral fails to describe properly transport processes which depend on the zeroth and second moments of the distribution function, i.e., the average energy and the heating of the plasma.[17] This proves to be an important point when the limits of validity of the linearized theory are considered.

A solution to Eq. (A1) may be written formally as an integral over particle trajectories.[10]

$$f(\mathbf{x},\mathbf{v},t)=\int_{T_0}^{t}-\nu_c[\delta(\mathbf{x}-\mathbf{x}_{cl})\delta(\mathbf{v}-\mathbf{v}_{cl})f_0(\mathbf{v}_0)e^{-\nu_c(t-t_0)}]d\mathbf{x}_0d\mathbf{v}_0dt_0+f_0(\mathbf{v})e^{-\nu_c(t-T_0)}, \quad \text{(A2)}$$

where T_0 is the time at which the external field is turned on.

In this case $\mathbf{x}_{cl}(\mathbf{x}_0,\mathbf{v}_0,t,t_0)$ and $\mathbf{v}_{cl}(\mathbf{x}_0,\mathbf{v}_0,t,t_0)$ are the solutions of the equations of motion of a particle of charge $-e$ and mass m in a static magnetic field[18] $\mathbf{B}_0$ under the action of a field $\mathbf{E}\exp i(kz-\omega t)$ (assumed known) with initial displacement and velocities $\mathbf{x}_0$ and $\mathbf{v}_0$ at time t_0. The integrals over $\mathbf{x}_0$ and $\mathbf{v}_0$ are easily performed and the resultant $f(\mathbf{x},\mathbf{v},t)$ is given as a single integral

$$f(\mathbf{x},\mathbf{v},t)=\int_{T_0}^{t}\begin{vmatrix}\partial x_{cl}/\partial x_0 & \partial x_{cl}/\partial v_0\\ \partial v_{cl}/\partial x_0 & \partial v_{cl}/\partial v_0\end{vmatrix}^{-1}\nu_c f_0(\mathbf{v}_0)e^{-\nu_c(t-t_0)}dt_0+f_0(\mathbf{v})\exp[-\nu_c(t-T_0)] \quad \text{(A3)}$$

The quantities $\mathbf{x}_0$ and $\mathbf{v}_0$ are evaluated along the actual path of the particle. The solution of the equations of motion,

$$m(d\mathbf{v}/dt)=-e[\mathbf{E}+\mathbf{v}\times\mathbf{B}_0], \quad \text{(A4)}$$

may be carried out in straightforward manner. We find, for a field of the form $\mathbf{E}=Ee^{i(kz-\omega t)}[\hat{u}_x\pm i\hat{u}_y]$ (where $\hat{u}_x$ and $\hat{u}_y$ are unit vectors in the x and y directions) that

$$v_x(t)=\Gamma(t-t_0)+v_x(t_0)\cos\omega_b(t-t_0)\mp v_y(t_0)\sin\omega_b(t-t_0), \quad \text{(A5a)}$$

$$v_y(t)=i\Gamma(t-t_0)+v_x(t_0)\sin\omega_b(t-t_0)\pm v_y(t_0)\cos\omega_b(t-t_0), \quad \text{(A5b)}$$

where

$$\omega_b=eB_0/m;\quad \omega_d=kv_z(t_0),$$

and

$$\Gamma(t-t_0)=+\frac{ieE}{m}\frac{1}{(\omega-\omega_d-\omega_b)}\times[e^{-i\omega_b t}e^{-i(\omega-\omega_d-\omega_b)t_0}-e^{-i(\omega-\omega_b)t}]. \quad \text{(A6)}$$

Equations (A5a) and (A5b) are easily inverted, yielding

$$v_x^2(t_0)+v_y^2(t_0)+v_z^2(t_0)=(v_x^2+v_y^2+v_z^2)-2(v_x\pm iv_y)\Gamma(t-t_0). \quad \text{(A7)}$$

For $\omega\neq\omega_d+\omega_b$, the term $\Gamma(t-t_0)$, being linear in E, may be treated as a small perturbation. A subsequent expansion in powers of E of the distribution function f_0 in Eq. (A3) yields exactly the results obtained by linearizing the Boltzmann equation. The expansion corresponds to evaluating the trajectory integral (A3) not along the actual trajectory of the particle, but along its unperturbed path.

For $\omega=\omega_d+\omega_b$, $\Gamma(t-t_0)$ increases linearly with time. In this case it is incorrect to treat it as a small perturbation if elapsed time in (A3) is allowed to run over a sufficiently large range. Of course, the dominant contributions to the integral in (A3) arise from times less than or equal to $1/\nu_c$. Thus, the second term on the right-hand side of Eq. (A7) can still be treated as a perturbation provided ν_c is sufficiently large, i.e.,

$$\nu_c>eE/m\langle v^2\rangle^{1/2}, \quad \text{(A8)}$$

where $\langle v^2\rangle$ is some proper average of the square of the thermal velocity. The above inequality implies that the effective static electric field which the resonant electrons experience cannot be allowed to act for times long compared with the time it takes to heat those electrons. Under such conditions the linearized approximation is not valid. This breakdown of the linearization is to be contrasted with the failure of the linear theory in the description of steady-state longitudinal plasma oscillations. For longitudinal oscillations it is a problem of phase coherence.[10,19] Here it is a heating problem which is not adequately treated by the momentum relaxation term used in the Boltzmann equation. It is impossible, using a relaxation term of the form employed in Eq. (A1), to find the correct first-order corrections brought about by the nonlinearity. However, it is possible to set the limits of linear theory and these are given by Eq. (A8).

[17] W. P. Allis, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. XXI, p. 410.

[18] We neglect the ac component of the magnetic field. This is tantamount to neglecting radiation pressure.

[19] J. Dawson, Phys. Fluids, 4, 869 (1961).