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Quasi-Classical Theory of a Relativistic Spinning Electron*

RALPH SCHILLER
Stevens Institute of Technology, Hoboken, New Jersey
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The quasi-classical spin theory is a modified Hamilton-Jacobi theory for a classical relativistic dipole whose space-time trajectory is coupled to its spin motion. The equations of this quasi-classical theory are almost identical with the second-order (squared) Dirac equation, and, in fact, form a new WKB approximation to the second-order Dirac equation. We quantize this theory by requiring that a classical spinor "wave function" be continuous and single valued, and that it satisfy an eigenvalue equation. We find for a particle in a uniform magnetic field, and in a Coulomb field, that this quasi-classical theory predicts the same energy and angular momentum eigenvalues as the Dirac theory. The quasi-classical theory is invariant under Lorentz transformation, spin rotations, and charge conjugation. The spinor wave functions are form invariant with respect to arbitrary canonical transformations.

I. INTRODUCTION

In three previous articles[1] we have shown how classical methods could be used to construct WKB solutions to the Schrödinger and Pauli equations. In this present article we prove that similar techniques are useful in finding these approximate solutions to the Dirac equation.

Our methods differ from other WKB approximations[2] in that we do not find these approximate wave functions by the usual asymptotic expansions of either the first- or second-order Dirac equations. Rather, we start with a definite classical model of a spinning electron which, in its mathematical formulation, makes use of a classical spinor "wave function," and then show that this spinor satisfies an equation differing from the second-order Dirac equation by terms that depend on $\hbar^2$. Eigenvalues associated with the classical constants of the motion are found by the traditional methods of the WKB approximation, and the requirement that the classical wave functions satisfy a given eigenvalue equation. For the case of an electron in a uniform magnetic field, and again in a Coulomb field, we derive the energy and angular momentum eigenvalues of the original Dirac equation.

Our WKB solutions to the Dirac equation depend on the classical model we choose at the outset. Conversely, a given asymptotic expansion in powers of $\hbar$ of a solution of the Dirac equation may imply a specific classical model for the electron.

Pauli[3] was the first to find a particular WKB approximation to the Dirac equation. The classical model implied by his WKB theory is that of a relativistic spinning particle whose translational motion is unaffected by its spin motion. The justification for this model is given by Pauli[4] as follows: In contrast to the mass and charge of the free electron, the spin is not a well-defined classical concept. It proves impossible to measure the intrinsic magnetic moment of a free electron by coupling the particle path to the spin motion, because any such measurement of the magnetic moment brings into play diffraction effects which preclude its observation. Thus, while we may picture an individual electron as having an intrinsic angular momentum, and we may actually calculate (classically) the influence that the spin has on the trajectory of the electron, the quantum theory of measurement rules out any experimental confirmation of this effect. The spin should then be excluded from any equations for the trajectory of a particle arising in the WKB approximation, since the WKB theory is the classical limit of the original wave theory. If one of the equations of the WKB theory has the form of the relativistic Hamilton-Jacobi equation, no spin-dependent terms should be permitted to appear in this equation, since such terms have no precise meaning in classical dynamics.

Although we agree with Pauli concerning the measurement of the free electron's magnetic moment, our view of the WKB approximation is different from his for three distinct reasons. In the first place, in the WKB theory one should seek the best possible asymptotic solution of the wave equation, and it is immaterial whether or not all aspects of the classical model are verifiable through experimental analysis. Our main requirements are rather that the WKB wave functions provide accurate particle densities everywhere except at the classical turning points, and that these wave functions give rise to experimentally confirmable eigenvalues and transition probabilities. In principle then, any classical model may be implied by the WKB theory, as long as the WKB wave functions are in close approximation to the true wave functions.

Secondly, since the WKB approximation is so closely tied to our classical view of nature, one should expect that, under specific experimental conditions, the classical elements of the theory should be subject to qualitative verification. For example, if we include the spin among the classical attributes of the electron, it is desirable that this quality manifest itself in some, but

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[1] R. Schiller, Phys. Rev. **125**, 1100, 1109, 1116 (1962). The articles are referred to as A, B, and C, respectively.

[2] W. Pauli, Helv. Phys. Acta **5**, 179 (1932); R. J. Bessey, thesis, University of Michigan, 1942 (unpublished); R. H. Good, Jr., Phys. Rev. **90**, 131 (1953); **94**, 931 (1954).

[3] W. Pauli, Helv. Phys. Acta **5**, 179 (1932).

[4] In his writings on this question, Pauli has followed the arguments of N. Bohr, J. Chem. Soc. 349 (1932).

not necessarily all, classically describable experiments. Thus, we are willing to accept spin as a classical concept even though no measurement is possible of the magnetic moment of the free electron. A Stern-Gerlach experiment for a silver atom provides sufficient justification for our classical image of the electron's spin.

Finally, one should also require that the classical theory, at least in its Hamilton-Jacobi form, contain the significant formal features present in the corresponding quantum theory, such as the appropriate transformation properties, conservation laws, etc.

For these reasons we adopt a WKB expansion different from Pauli's; one which originates from a classical model in which there is mutual coupling between the spin and the trajectory of the electron. We hope in this way to find WKB wave functions that are superior approximations to the true wave solutions, and at the same time to reproduce in the classical theory many of the qualitative features of the quantum theory.

Our WKB approximation runs counter to another well-established principle of the quantum theory which holds that, in the limit as h vanishes, the quantum theory goes over into classical mechanics. Within the confines of the nonrelativistic Schrödinger theory, there is general agreement as to how and when this rule is to be applied. Matters are more complicated in a quantum theory of a spinning electron. For in such a theory, Planck's constant plays a dual role. Its appearance confirms, as in the Schrödinger theory, the wavelike character of the electron. On the other hand, it is the parameter which describes the electron's intrinsic magnetic moment. It is h in this latter capacity which appears in our classical theory. We believe that the h associated with the spin should be retained in the classical theory for the three reasons we have already cited. The presence of h in this form in the classical theory in no wise leads to electron diffraction, quantized eigenvalues, or any other effect that is peculiar to a wave theory.

II. CLASSICAL RELATIVISTIC DIPOLE

As a model for a relativistic dipole we adopt the simplest possible generalization of the nonrelativistic dipole which we treated in C.[1] Relativity requires that an electric dipole moment appear together with the magnetic dipole moment, so that together they act as a generalized entity which we call the relativistic dipole. It is customary to demand that the electric dipole moment vanish in the rest frame of the electron, a requirement which permits one to express the electric dipole moment in terms of the magnetic dipole moment and the electron's velocity.[5] However, in this paper we shall forego this requirement and treat the electric dipole moment as independent of the magnetic dipole moment.

[5] H. A. Kramers, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1957), Sec. 57.

In nonrelativistic mechanics, a classical magnetic dipole is described by a spin vector $\mathbf{s}$, whose magnitude is a constant of the motion and whose dynamics is governed by the equation

$$d\mathbf{s}/dt=-\alpha(\mathbf{s}\times\mathbf{B}),\quad \alpha=e/mc. \tag{1}$$

In Eq. (1), $\mathbf{B}$ is the external magnetic field, and $-e$ and m are, respectively, the charge and mass of the dipole (electron).

In our relativistic model we shall represent the dipole by the complex vector $\mathbf{\Sigma}=\mathbf{M}+i\mathbf{N}$,[6] where $\mathbf{M}$ and $\mathbf{N}$ are the magnetic and electric dipole moments, respectively. Alternatively, we could define the dipole in terms of the components of an antisymmetric tensor $M_{\mu\nu}$, where the space parts of the tensor, M_{rs}, represent the magnetic dipole moment $\mathbf{M}$, and the mixed space-time components, M_{r4}, the electric dipole moment $i\mathbf{N}$.

The relativistic equations of motion for the dipole are found by replacing $\mathbf{s}$ by $\mathbf{\Sigma}$ and $\mathbf{B}$ by $\mathbf{F}=\mathbf{B}-i\mathbf{E}$, so that (1) becomes

$$d\mathbf{\Sigma}/d\tau=-\alpha(\mathbf{\Sigma}\times\mathbf{F}). \tag{2}$$

τ is the real proper time and $\mathbf{E}$ the external electric field. The two real equations of motion in (2) are

$$d\mathbf{M}/d\tau=-\alpha(\mathbf{M}\times\mathbf{B}+\mathbf{N}\times\mathbf{E}), \tag{3a}$$

$$d\mathbf{N}/d\tau=-\alpha(\mathbf{N}\times\mathbf{B}-\mathbf{M}\times\mathbf{E}). \tag{3b}$$

For arbitrary external fields, these equations admit the two constants of the motion, $\mathbf{M}^2-\mathbf{N}^2$ and $\mathbf{M}\cdot\mathbf{N}$; consequently, they are four independent equations for four unknown quantities.

As shown by Kramers,[7] Eqs. (3) may be placed in canonical form by introducing the spin Hamiltonian,

$$\begin{aligned}H_{\mathrm{sp}}&=\tfrac{1}{2}\alpha(\mathbf{\Sigma}\cdot\mathbf{F}+\text{c.c.})\\&=\tfrac{1}{2}\alpha\{[(F_x\sin\eta+F_y\cos\eta)(\Sigma^2-\xi^2)^{\frac{1}{2}}+\xi F_z]+\text{c.c.}\},\end{aligned} \tag{4}$$

with $-\eta/2=-\frac{1}{2}\arctan(\Sigma_x/\Sigma_y)$ and its complex conjugate (c.c.) acting as canonical coordinates, and $\xi=\Sigma_z$ and its complex conjugate acting as the momenta conjugate to $-\eta/2$ and $-\eta^*/2$. The real part of the spin coordinate is the azimuthal angle of the spin orientation as measured from the y axis in the laboratory frame of reference. The momentum variable is the z component of the complex spin vector. Hamilton's equations of motion for the complex variables η and ξ are

$$\frac{\partial H_{\mathrm{sp}}}{\partial\xi}=\frac{d}{d\tau}\left(-\frac{\eta}{2}\right),\qquad \frac{\partial H_{\mathrm{sp}}}{\partial(-\eta/2)}=-\frac{d\xi}{d\tau}; \tag{5a}$$

$$\frac{\partial H_{\mathrm{sp}}}{\partial\xi^*}=\frac{d}{d\tau}\left(-\frac{\eta^*}{2}\right),\qquad \frac{\partial H_{\mathrm{sp}}}{\partial(-\eta^*/2)}=-\frac{d\xi^*}{d\tau}. \tag{5b}$$

[6] Reference 5, p. 229.

[7] H. A. Kramers, Verhandl. Zeeman Jubil. 403 (1935). This article is reproduced in the *Collected Scientific Papers of H. A. Kramers* (North-Holland Publishing Company, Amsterdam, 1950), p. 687.

These are the four equations for the four independent parts of the tensor $M_{\mu\nu}$.

The Eqs. (3) are also derivable from a variational principle, with a spin Lagrangian defined in the usual way:

$$L_{\mathrm{sp}}=-H_{\mathrm{sp}}+\xi\frac{d}{d\tau}\left(-\frac{\eta}{2}\right)+\xi^*\frac{d}{d\tau}\left(-\frac{\eta^*}{2}\right). \tag{6}$$

We shall now show that the equations of motion, (3), may be rewritten in terms of a relativistic spinor. We note again, as we have in the paragraph following Eq. (1), that the real and imaginary parts of the complex spin vector Σ may be identified with the components of the antisymmetric tensor $M_{\mu\nu}$. We define such a tensor in terms of the relativisitc spinor ϕ,

$$\begin{aligned} M_{\mu\nu}&=-i\bar{\phi}\gamma_\mu\gamma_\nu\phi \quad \text{for} \quad \mu\neq\nu,\\ M_{\mu\nu}&=0 \qquad\qquad \text{for} \quad \mu=\nu, \end{aligned} \tag{7}$$

where ϕ is given as[8]

$$\phi=|\Sigma|^{\frac{1}{2}}\begin{bmatrix} \cos(\theta'/2)\exp(i\eta/2)\\ i\sin(\theta'/2)\exp(-i\eta/2)\\ \cos(\theta'^*/2)\exp(i\eta^*/2)\\ i\sin(\theta'^*/2)\exp(-i\eta^*/2)\end{bmatrix}$$

and $|\Sigma|\cos\theta'=\xi$. $|\Sigma|$ is a constant and is the absolute value of the total spin, $|\Sigma|=(M_{\mu\nu}M_{\mu\nu}/2)^{\frac{1}{2}}$.

In practice, however, it proves more convenient to introduce another angle θ'', where $\xi=(\hbar/2)\cos\theta''=|\Sigma|\cos\theta'$. h is later to be identified with Planck's constant. In terms of the angle θ'', the spinor ϕ is

$$\phi=(\tfrac{1}{2}\hbar)^{1/2}\begin{bmatrix} \cos(\theta''/2)\exp(i\eta/2)\\ i\sin(\theta''/2)\exp(-i\eta/2)\\ \cos(\theta''^*/2)\exp(i\eta^*/2)\\ i\sin(\theta''^*/2)\exp(-i\eta^*/2)\end{bmatrix}. \tag{8}$$

The γ_μ in (7) are the matrices

$$\gamma_s=\begin{pmatrix} 0 & i\sigma_s\\ -i\sigma_s & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix},$$

and $\bar{\phi}=\phi^\dagger\gamma_4$.

The Lagrangian (6), in terms of the spinor ϕ, is

$$L_{\mathrm{sp}}=\tfrac{1}{2}\left(\bar{\phi}\frac{d\phi}{d\tau}-\frac{d\bar{\phi}}{d\tau}\phi\right)+\tfrac{1}{4}\alpha F_{\mu\nu}\bar{\phi}\gamma_\mu\gamma_\nu\phi, \tag{9}$$

where the antisymmetric tensor $F_{\mu\nu}$ represents the electromagnetic field, $F_{rs}\to\mathbf{B}$, $F_{r4}\to -i\mathbf{E}$. Since we are using an imaginary time coordinate we need not distinguish between covariant and contravariant components of vectors and tensors, when our coordinates are Cartesian.

If we vary ϕ and $\bar{\phi}$ as independent variables in (9), we get the following equations:

$$d\phi/d\tau=-\tfrac{1}{2}i\alpha(\mathbf{B}\cdot\boldsymbol{\sigma}'-i\mathbf{E}\cdot\boldsymbol{\alpha})\phi, \tag{10a}$$

$$d\bar{\phi}/d\tau=\tfrac{1}{2}i\alpha\bar{\phi}(\boldsymbol{\sigma}'\cdot\mathbf{B}-i\boldsymbol{\alpha}\cdot\mathbf{E}). \tag{10b}$$

It is not difficult to show that Eqs. (10) are identical with Eqs. (3), when we identify $\boldsymbol{\Sigma}$ with $M_{\mu\nu}$.

If we multiply Eq. (10a) by $i\hbar$ we get

$$i\hbar(d\phi/d\tau)=(e\hbar/2mc)(\mathbf{B}\cdot\boldsymbol{\sigma}'-i\mathbf{E}\cdot\boldsymbol{\alpha})\phi, \tag{11}$$

with

$$\boldsymbol{\sigma}'=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & \boldsymbol{\sigma}\end{pmatrix} \quad\text{and}\quad \boldsymbol{\alpha}=\begin{pmatrix}\boldsymbol{\sigma} & 0\\ 0 & -\boldsymbol{\sigma}\end{pmatrix}.$$

In complete analogy with the nonrelativistic theory of C,[1] we may quantize this classical dipole theory by requiring that ϕ satisfy the eigenvalue equation

$$i\hbar(d\phi/d\tau)=E_{\mathrm{sp}}\phi. \tag{12}$$

For a constant magnetic field along the z axis we find, as in the nonrelativistic theory, that $E_{\mathrm{sp}}=\pm e\hbar B/2mc$. From (4) we have $H_{\mathrm{sp}}=E_{\mathrm{sp}}=eBM_z/mc$, so that the requirement (12) has led to the quantization of the magnetic dipole moment in the z direction, $M_z=\pm\hbar/2$.

III. PROPER TIME FORMULATION OF CLASSICAL MECHANICS

Since we expect to make use of a proper time formulation of the Dirac theory, we should first like to discuss some aspects of this formalism in classical relativistic dynamics.[9]

The relativistic Hamilton-Jacobi equation is

$$\mathcal{H}=\frac{1}{2m}\left[\left(\nabla S+\frac{e}{c}\mathbf{A}\right)^2-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}-e\phi\right)^2+m^2c^2\right]=0. \tag{13}$$

This equation may be put in the form of the nonrelativistic Hamilton-Jacobi equation by requiring that S satisfy the equation

$$\partial S/\partial\tau+\mathcal{H}=0, \tag{14}$$

where the variable τ (the proper time) is to be considered as a fifth independent variable. Solutions of Eq. (14) become solutions of the relativistic Hamilton-Jacobi equation when the following condition is imposed on S:

$$\partial S/\partial\tau=0. \tag{15}$$

There are an infinite number of solutions to (14). For any particular solution, the condition (15) yields a specific relation among the space-time variables, the proper time, and the constants appearing in that solution. For example, if we wish to describe an ensemble of particles which originates from the space-time point $x_0{}^\mu$ at the proper time $\tau=0$, S must be chosen as the function

$$S=S(x^\mu,x_0{}^\mu,\tau). \tag{16}$$

[8] ϕ, in the form given below, transforms as a spinor in the representation where $\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$.

[9] See V. Fock, Physik. Z. Sowjetunion **12**, 404 (1937).

The condition (15) then fixes τ as a function of two points in space-time.

On the other hand, for mechanical problems where energy and angular momentum are conserved, one has

$$S=S(E,L,L_z,K,x^\mu,\tau), \tag{17}$$

where E is the energy, L the total angular momentum, L_z the z component of the angular momentum, and K a separation constant for the variable τ.

The condition (15) then tells us that, for this ensemble, the constant K vanishes.

In the proper time formulation of the relativistic WKB theory, we find that a condition similar to (15) must be imposed on the classical wave functions.

IV. THE PROPER TIME IN A GENERALIZED HAMILTON-JACOBI THEORY

In this section we consider the full motion (both rotation and translation) of a relativistic dipole. We show how the dynamical equations may be written as a generalized Hamilton-Jacobi theory in which the proper time is introduced as an independent variable.

Assume that the motion is defined by the following equations:

$$\frac{d}{d\tau}(mV_\mu)=-\frac{e}{c}F_{\mu\nu}V_\nu-\frac{e}{2mc}M_{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial x^\mu}; \tag{18a}$$

$$\frac{1}{2}\frac{d\xi}{d\tau}=\frac{\partial H_{\mathrm{sp}}}{\partial\eta},\quad \frac{1}{2}\frac{d\eta}{d\tau}=-\frac{\partial H_{\mathrm{sp}}}{\partial\xi}; \tag{18b}$$

$$\frac{1}{2}\frac{d\xi^*}{d\tau}=\frac{\partial H_{\mathrm{sp}}}{\partial\eta^*},\quad \frac{1}{2}\frac{d\eta^*}{d\tau}=-\frac{\partial H_{\mathrm{sp}}}{\partial\xi^*}; \tag{18c}$$

where $F_{\rho\sigma}=\partial A_\sigma/\partial x^\rho-\partial A_\rho/\partial x^\sigma$, $A_\rho=(\mathbf{A},i\phi)$, $V_\mu=(\mathbf{v}\gamma, ic\gamma)$, $M_{12}=M_3$, $M_{34}=iN_3$, etc., and $H_{\mathrm{sp}}=(e/2mc)F_{\rho\sigma}M_{\rho\sigma}$.

The classical second-order proper-time Dirac equation is the Hamilton-Jacobi theory of the coupled set of equations, (18), when this theory is expressed in terms of the dynamical variables of the translational motion alone.

Before we derive this Hamilton-Jacobi theory, we shall have to say a few words about one of the constants of the motion predicted by the dynamical equations, (18). If we multiply (18a) by V_μ, and make use of (18b, 18c), we find that the spin energy, $H_{\mathrm{sp}}=(e/2mc)F_{\rho\sigma}M_{\rho\sigma}$, is a constant of the motion. It is well known that, in general, no such constant of the motion exists in the Dirac theory. Thus, our model at the outset seems to be in contradiction with the theory to which, supposedly, it is in correspondence. However, a minor modification will change the equations in such a way that the classical theory will be consistent with the quantum theory.

We have until now assumed that the mass which appears in (18) is the rest mass, and thus a constant of the motion. We shall now permit the mass appearing in (18) to be a function of the proper time, so that (18a) reads

$$\frac{dm}{d\tau}V_\mu+m\frac{dV_\mu}{d\tau}=-\frac{e}{c}F_{\mu\nu}V_\nu-\frac{e}{2mc}M_{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial x^\mu}. \tag{18a$'$}$$

If we now multiply the above equation by mV_μ and make use of (18b, c), we find the following constant of the motion,

$$\tfrac{1}{2}m_0^2c^2\equiv\tfrac{1}{2}m^2c^2-(e/2c)F_{\rho\sigma}M_{\rho\sigma}. \tag{19a}$$

By introducing a new parameter ϑ in place of the proper time τ, the variable mass $m(\tau)$ can be made to disappear from the dynamical equations, (18). If we write $(d\vartheta/d\tau)=m/m_0$, the dynamical equations (18) become

$$m_0\frac{d^2x^\mu}{d\vartheta^2}=-\frac{e}{c}F_{\mu\nu}\frac{dx^\nu}{d\vartheta}-\frac{e}{2m_0c}M_{\rho\sigma}\frac{\partial F_{\rho\sigma}}{\partial x^\mu}; \tag{18a$'$}$$

$$\frac{1}{2}\frac{d\xi}{d\vartheta}=\frac{\partial H_{\mathrm{sp}}'}{\partial\eta},\quad \frac{1}{2}\frac{d\eta}{d\vartheta}=-\frac{\partial H_{\mathrm{sp}}'}{\partial\xi}; \tag{18b$'$}$$

$$\frac{1}{2}\frac{d\xi^*}{d\vartheta}=\frac{\partial H_{\mathrm{sp}}'}{\partial\eta^*},\quad \frac{1}{2}\frac{d\eta^*}{d\vartheta}=-\frac{\partial H_{\mathrm{sp}}'}{\partial\xi^*}; \tag{18c$'$}$$

where $H_{\mathrm{sp}}'=(e/2m_0c)F_{\rho\sigma}M_{\rho\sigma}$.

These equations may be derived from the Hamiltonian

$$H=\frac{1}{2m_0}\left[\left(P_\mu+\frac{e}{c}A_\mu\right)^2+\frac{e}{c}F_{\rho\sigma}M_{\rho\sigma}+m_0^2c^2\right]=0, \tag{19b}$$

where $P_\mu=m_0(dx^\mu/d\vartheta)-(e/c)A_\mu$. This Hamiltonian vanishes by virtue of (19a).

We are now prepared to find the generalized Hamilton-Jacobi equations by the same means employed in C.[1] We assume that the variables V_μ and $M_{\mu\nu}$ are continuous field functions of the proper time ϑ and the coordinates x^μ. We write the operator $d/d\vartheta$ in (18a′) as $\partial/\partial\vartheta+(dx^\mu/d\vartheta)(\partial/\partial x^\mu)$, differentiate H of (19b) with respect to the coordinates x^μ, add the two equations together, and find the following relation:

$$\begin{aligned}0=&\frac{\partial}{\partial\vartheta}\left(m_0\frac{dx^\mu}{d\vartheta}-\frac{\xi}{2}\frac{\partial\eta}{\partial x^\mu}-\frac{\xi^*}{2}\frac{\partial\eta^*}{\partial x^\mu}-\frac{e}{c}A_\mu\right)\\&+\frac{\partial}{\partial x^\mu}\left(H+\frac{\xi}{2}\frac{\partial\eta}{\partial\vartheta}+\frac{\xi^*}{2}\frac{\partial\eta^*}{\partial\vartheta}\right)\\&+\left[\frac{\partial}{\partial x^\mu}\left(m_0\frac{dx^\nu}{d\vartheta}-\frac{e}{c}A_\nu-\frac{\xi}{2}\frac{\partial\eta}{\partial x^\nu}-\frac{\xi^*}{2}\frac{\partial\eta^*}{\partial x^\nu}\right)\right.\\&\left.-\frac{\partial}{\partial x^\nu}\left(m_0\frac{dx^\mu}{d\vartheta}-\frac{e}{c}A_\mu-\frac{\xi}{2}\frac{\partial\eta}{\partial x^\mu}-\frac{\xi^*}{2}\frac{\partial\eta^*}{\partial x^\mu}\right)\right]\frac{dx^\nu}{d\vartheta}.\end{aligned} \tag{20}$$

Equation (20) may be satisfied by the equations

$$\frac{\partial S}{\partial \vartheta}+\tfrac{1}{2}\xi\frac{\partial \eta}{\partial \vartheta}+\tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial \vartheta}+H=0, \tag{21a}$$

$$\frac{\partial S}{\partial x^\nu}=m_0\frac{dx^\nu}{d\vartheta}-\frac{e}{c}A_\nu-\tfrac{1}{2}\xi\frac{\partial \eta}{\partial x^\nu}-\tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial x^\nu}. \tag{21b}$$

Equation (21a) is a generalized Hamilton-Jacobi equation, and (21b) a generalized definition of momentum,

$$P_\mu=m_0\frac{dx^\mu}{d\vartheta}-\frac{e}{c}A_\mu=\frac{\partial S}{\partial x^\mu}+\frac{\xi}{2}\frac{\partial \eta}{\partial x^\mu}+\frac{\xi^*}{2}\frac{\partial \eta^*}{\partial x^\mu}.$$

The dynamical equations, (18), are now replaced by the three sets of coupled equations:

$$\frac{\partial S}{\partial \vartheta}+\frac{\xi}{2}\frac{\partial \eta}{\partial \vartheta}+\frac{\xi^*}{2}\frac{\partial \eta^*}{\partial \vartheta}+\frac{1}{2m_0}\times\left[m_0^2\left(\frac{dx^\mu}{d\vartheta}\right)^2+\frac{e}{c}F_{\rho\sigma}M_{\rho\sigma}+m_0^2c^2\right]=0; \tag{22a}$$

$$\frac{1}{2}\frac{d\xi}{d\vartheta}=\frac{\partial H_{\mathrm{sp}}'}{\partial \eta},\quad \frac{1}{2}\frac{d\eta}{d\vartheta}=-\frac{\partial H_{\mathrm{sp}}'}{\partial \xi}; \tag{22b}$$

$$\frac{1}{2}\frac{d\xi^*}{d\vartheta}=\frac{\partial H_{\mathrm{sp}}'}{\partial \eta^*},\quad \frac{1}{2}\frac{d\eta^*}{d\vartheta}=-\frac{\partial H_{\mathrm{sp}}'}{\partial \xi^*}. \tag{22c}$$

We can show that these equations lead to an equation of continuity,

$$\frac{\partial \rho}{\partial \vartheta}+\frac{\partial}{\partial x^\mu}\left(\rho\frac{dx^\mu}{d\vartheta}\right)=0, \tag{23}$$

where the density ρ is the determinant

$$\rho\equiv\left\|\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_\nu}+\frac{\xi}{2}\frac{\partial \eta}{\partial \alpha_\nu}+\frac{\xi^*}{2}\frac{\partial \eta^*}{\partial \alpha_\nu}\right)\right\|,$$

and the constants α_ν are four constants of the motion that appear in the solutions for S, ξ, ξ^*, η, and η^*. We omit the proof since it is essentially the same as that given for a nonrelativistic density in C.[1]

V. CLASSICAL DIRAC THEORY

We now show that the six equations in (22) and (23) may be written in a form similar to the second-order proper-time Dirac equation.

We first observe that the six equations may be derived from a variational principle where the Lagrangian is

$$L=\int \rho\left\{\frac{\partial S}{\partial \vartheta}+\frac{\xi}{2}\frac{\partial \eta}{\partial \vartheta}+\frac{\xi^*}{2}\frac{\partial \eta^*}{\partial \vartheta}+\frac{1}{2m_0}\times\left[\left(m_0\frac{dx^\mu}{d\vartheta}\right)^2+\frac{e}{c}F_{\rho\sigma}M_{\rho\sigma}+m_0^2c^2\right]\right\}d^3x, \tag{24}$$

and the variations are carried out with respect to the independent variables S, ρ, ξ, $-\eta/2$, ξ^*, and $-\eta^*/2$.

We introduce the four-component spinor ψ,

$$\psi=R\exp(iS/\hbar)\begin{pmatrix}\cos(\theta''/2)\exp(i\eta/2)\\ i\sin(\theta''/2)\exp(-i\eta/2)\\ \cos(\theta''^*/2)\exp(i\eta^*/2)\\ i\sin(\theta''^*/2)\exp(-i\eta^*/2)\end{pmatrix}, \tag{25}$$

where $R=\rho^{\frac{1}{2}}$ and $(\cos\theta'')(\hbar/2)=\xi$. If we substitute this spinor in the Lagrangian

$$L=\int d^3x\left\{\frac{\hbar}{2}\left(\bar\psi\frac{\partial\psi}{\partial\vartheta}-\frac{\partial\bar\psi}{\partial\vartheta}\psi\right)+\frac{1}{2m_0}\left[\left(i\hbar\frac{\partial\bar\psi}{\partial x^\mu}-\frac{e}{c}\bar\psi A_\mu\right)\left(-i\hbar\frac{\partial\psi}{\partial x^\mu}-\frac{e}{c}A_\mu\psi\right)+m_0^2c^2\bar\psi\psi\right]-\frac{e\hbar}{2c}\bar\psi\gamma_\mu\gamma_\nu\psi F_{\mu\nu}-\frac{\hbar^2}{16m_0}\left[\frac{\partial}{\partial x^\rho}(\bar\psi\gamma_\mu\gamma_\nu\psi)\times\frac{\partial}{\partial x^\rho}(\bar\psi\gamma_\mu\gamma_\nu\psi)\right]\frac{1}{\bar\psi\psi}\right\}, \tag{26}$$

we find that (24) and (26) are identical. However, if we consider the "wave function" ψ in (26) as the unknown function in the variational principle, we find that ψ satisfies the equation

$$i\hbar\frac{\partial\psi}{\partial\vartheta}=\frac{1}{2m_0}\left\{\left(-i\hbar\frac{\partial}{\partial x^\mu}+\frac{e}{c}A_\mu\right)^2\psi+m_0^2c^2\psi-\frac{ie\hbar}{2c}\gamma_\mu\gamma_\nu\psi F_{\mu\nu}-\frac{\hbar^2}{8}\frac{\delta}{\delta\bar\psi}\times\left\{\int d^3x\left[\frac{\partial}{\partial x^\rho}(\bar\psi\gamma_\mu\gamma_\nu\psi)\frac{\partial}{\partial x^\rho}(\bar\psi\gamma_\mu\gamma_\nu\psi)\right]\frac{1}{\bar\psi\psi}\right\}. \tag{27}$$

In the final terms in both Eq. (26) and Eq. (27), we sum over components for which $\mu\neq\nu$. Equation (27) is identical with the second-order proper-time Dirac equation, except for the last term.

VI. GENERALIZED HAMILTON-JACOBI THEORY WITHOUT PROPER TIME

In actual fact, we are not interested in solutions to Eq. (27). Rather, we seek wave functions which satisfy that equation with vanishing left-hand side,

$$0=\left(-i\hbar\frac{\partial}{\partial x^\mu}+\frac{e}{c}A_\mu\right)^2\psi+m_0^2c^2\psi-\frac{ie\hbar}{2c}\gamma_\mu\gamma_\nu F_{\mu\nu}\psi-\frac{\hbar^2}{8}\frac{\delta}{\delta\bar\psi}\left\{\int d^3x\left[\frac{\partial}{\partial x^\rho}(\bar\psi\gamma_\mu\gamma_\nu\psi)\frac{\partial}{\partial x^\rho}(\bar\psi\gamma_\mu\gamma_\nu\psi)\right]\frac{1}{\bar\psi\psi}\right\}. \tag{28}$$

In the classical theory there are several ways in which such solutions can be generated from the solutions of

Eq. (27). For ensembles of particles whose state at $\vartheta=0$ is described by a four-dimensional delta function, Fock[10] has given the following rule: Replace ψ by ψ', where

$$\psi'=\int_C \psi d\vartheta, \tag{29}$$

and where the path of integration C is chosen so that the condition

$$\frac{\partial\psi'}{\partial\vartheta}=\int_C \frac{\partial\psi}{\partial\vartheta}d\vartheta=\psi\Big|_C=0 \tag{30}$$

is satisfied. Equation (30) is the analog to the requirement (15).

ψ' is then a solution of Eq. (28). Moreover, for these delta-function (point source) ensembles, there are many dynamical problems for which the last term on the right hand side of Eq. (27) vanishes. ψ' is then a solution of the second-order Dirac equation, and can be used to generate solutions of the first-order Dirac equation. This fact is of major significance in Feynman's "sum over paths" formulation of the Dirac theory, as well as Schwinger's "action principle" formulation of the same theory, and helps to explain why, for special ensembles, solutions to classical problems may be used to find solutions to corresponding problems in the Dirac theory.[11,12]

For arbitrary ensembles of particles, it is not always a trivial matter to find a path of integration for which $\partial\psi'/\partial\vartheta=0$, and so Fock's prescription may not prove useful for all the families of particles to be described in the quantum theory or its WKB approximation. Therefore, in this section we shall present a different method for finding solutions to Eq. (28).

Our starting point is the Hamiltonian, (19b). We go over to the Hamilton-Jacobi equation in the six-dimensional configuration space of the variables x^μ, η, and η^* by making the following substitutions in (19b):

$$P_\mu=\frac{\partial\mathcal{S}}{\partial x^\mu},\quad \frac{\xi}{2}=-\frac{\partial\mathcal{S}}{\partial\eta},\quad \frac{\xi^*}{2}=-\frac{\partial\mathcal{S}}{\partial\eta^*}. \tag{31}$$

The relativistic Hamilton-Jacobi equation for a particle with spin is then

$$\frac{1}{2m_0}\left[\left(\frac{\partial\mathcal{S}}{\partial x^\mu}+\frac{e}{c}A_\mu\right)^2+\frac{e}{c}F_{\rho\sigma}M_{\rho\sigma}+m_0^2c^2\right]=0. \tag{32}$$

A solution of (32) is of the form

$$\mathcal{S}=\mathcal{S}(x^\mu,\eta,\eta^*,\alpha^A), \tag{33}$$

where the α^A are five constants of integration, none of which is additive.

[10] Reference 9, p. 408.
[11] R. P. Feynman, Phys. Rev. **84**, 108 (1951), Appendix D.
[12] J. Schwinger, Phys. Rev. **82**, 664 (1951).

It is not difficult to show that five additional independent constants of the motion, β_A, may be found by differentiating $\mathcal{S}$ with respect to the α^A,

$$\beta_A=\partial\mathcal{S}/\partial\alpha^A. \tag{34}$$

We now solve two of the Eqs. (34) for the spin variables η and η^* and express them as functions of x^μ, α^A, and the two additional constants β_1 and β_2,

$$\begin{aligned}\eta&=\eta(x^\mu,\alpha^A,\beta_1,\beta_2),\\ \eta^*&=\eta^*(x^\mu,\alpha^A,\beta_1,\beta_2).\end{aligned} \tag{35}$$

η and η^*, as the functions given in (35), may now be substituted in the solution $\mathcal{S}$ so that

$$\mathcal{S}=S(x^\mu,\alpha^A,\beta_1,\beta_2). \tag{36}$$

Similarly, we may express the spin momenta ξ and ξ^* as functions of the same variables since

$$\tfrac{1}{2}\xi=-\partial\mathcal{S}/\partial\eta=\tfrac{1}{2}\xi(x^\mu,\alpha^A,\beta_1,\beta_2), \tag{37a}$$

$$\tfrac{1}{2}\xi^*=-\partial\mathcal{S}/\partial\eta^*=\tfrac{1}{2}\xi^*(x^\mu,\alpha^A,\beta_1,\beta_2). \tag{37b}$$

If we replace $\mathcal{S}$ in (32) by the S in (36), we find that

$$\frac{\partial\mathcal{S}}{\partial x^\mu}=\frac{\partial S}{\partial x^\mu}+\tfrac{1}{2}\xi\frac{\partial\eta}{\partial x^\mu}+\tfrac{1}{2}\xi^*\frac{\partial\eta^*}{\partial x^\mu}. \tag{38}$$

Equation (38) substituted in (32) yields

$$\frac{1}{2m_0}\left[\left(m_0\frac{dx^\mu}{d\vartheta}\right)^2+\frac{e}{c}F_{\rho\sigma}M_{\rho\sigma}+m_0^2c^2\right]=0, \tag{39}$$

where

$$m_0\frac{dx^\mu}{d\vartheta}=\frac{\partial S}{\partial x^\mu}+\tfrac{1}{2}\xi\frac{\partial\eta}{\partial x^\mu}+\tfrac{1}{2}\xi^*\frac{\partial\eta}{\partial x^\mu}-\frac{e}{c}A_\mu.$$

Equation (39) is identical with Eq. (22a), when we assume that the dependent variables are independent of the proper time. We have given this alternative derivation of (22a), since the procedure leading to Eq. (39) is an important practical technique for finding solutions to the quasi-classical equation.

The new Hamilton-Jacobi equation (39) must be supplemented by the spin equations

$$\frac{1}{2}\frac{\partial\xi}{\partial x^\mu}\frac{dx^\mu}{d\vartheta}=\frac{\partial H_{\mathrm{sp}}'}{\partial\eta},\quad \frac{1}{2}\frac{\partial\eta}{\partial x^\mu}\frac{dx^\mu}{d\vartheta}=-\frac{\partial H_{\mathrm{sp}}'}{\partial\xi}; \tag{40a}$$

$$\frac{1}{2}\frac{\partial\xi^*}{\partial x^\mu}\frac{dx^\mu}{d\vartheta}=\frac{\partial H_{\mathrm{sp}}'}{\partial\eta^*},\quad \frac{1}{2}\frac{\partial\eta^*}{\partial x^\mu}\frac{dx^\mu}{d\vartheta}=-\frac{\partial H_{\mathrm{sp}}'}{\partial\xi^*}. \tag{40b}$$

In this transformed Hamilton-Jacobi theory, the five equations, (39) and (40), must be solved simultaneously for the five dependent variables S, ξ, ξ^*, η, η^*.

Given a solution to (39) and (40), it is possible to find

a sixth dependent variable, $\bar{\rho}$, which satisfies a conservation law,[13]

$$\partial\bar{\rho}/\partial t+\nabla\cdot(\bar{\rho}\mathbf{v})=0, \tag{41}$$

where

$$\bar{\rho}=\left\|\frac{\partial}{\partial q_j}\left(\frac{\partial S}{\partial\alpha_i}+\tfrac{1}{2}\xi\frac{\partial\eta}{\partial\alpha_i}+\tfrac{1}{2}\xi^*\frac{\partial\eta^*}{\partial\alpha_i}\right)\right\|.$$

To preserve the relativistic character of the theory, it is necessary to introduce the new density ρ,

$$ic\bar{\rho}=\rho(dx^4/d\vartheta), \tag{42}$$

which satisfies the relativistic conservation law

$$\partial/\partial x^\mu(\rho dx^\mu/d\vartheta)=0. \tag{43}$$

We now introduce the four-component spinor

$$\psi=\rho^{\frac{1}{2}}e^{iS/\hbar}\begin{bmatrix}\cos(\theta''/2)\exp(i\eta/2)\\ i\sin(\theta''/2)\exp(-i\eta/2)\\ \cos(\theta''^*/2)\exp(i\eta^*/2)\\ i\sin(\theta''^*/2)\exp(-i\eta^*/2)\end{bmatrix}. \tag{44}$$

A calculation along the lines leading to Eq. (27) shows that ψ satisfies the equation

$$\left(-i\hbar\frac{\partial}{\partial x^\mu}+\frac{e}{c}A_\mu\right)^2\psi+m_0{}^2c^2\psi-\frac{ie\hbar}{2}\gamma_\mu\gamma_\nu\psi F_{\mu\nu}$$
$$-\frac{\hbar^2}{8}\frac{\delta}{\delta\bar{\psi}}\int\left\{\frac{\partial}{\partial x^\mu}(\bar{\psi}\gamma_\rho\gamma_\sigma\psi)\frac{\partial}{\partial x^\mu}(\bar{\psi}\gamma_\rho\gamma_\sigma\psi)\frac{1}{\bar{\psi}\psi}\right\}d^3x=0. \tag{45}$$

This method of constructing ψ completely avoids the introduction of the "proper time" ϑ.

VII. ELECTRON IN A UNIFORM MAGNETIC FIELD

In this section we show how one quantizes the classical proper time Dirac theory we presented in Sec. V. We illustrate the general method by treating the problem of an electron in a uniform magnetic field.

In the constant external field $\mathbf{B}=(0,0,B)$, the WKB solutions have the form[14]

$$\psi=R\Phi e^{iS/\hbar}. \tag{46}$$

S is a solution of the classical Hamilton-Jacobi equation,[15]

$$\frac{\partial S}{\partial\vartheta}+\frac{1}{2m_0}\left[\left(\frac{\partial S}{\partial x}-\frac{eB}{c}y\right)^2+\left(\frac{\partial S}{\partial y}\right)^2+\left(\frac{\partial S}{\partial z}\right)^2-\frac{1}{c^2}\left(\frac{\partial S}{\partial t}\right)^2+m_0{}^2c^2\right]=0. \tag{47}$$

R is the square root of the Van Vleck determinant,

$$R=\|\partial^2S/\partial x^\mu\partial\alpha^\nu\|^{\frac{1}{2}}, \tag{48}$$

and Φ is a four-component unit spinor, a solution of the ordinary differential equation

$$d\Phi/d\vartheta=-(ieB/2m_0c)\sigma_z'\Phi, \tag{49}$$

with

$$\sigma_z'=\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix},$$

and σ_z the Pauli spin matrix,

$$\sigma_z=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Equation (47) may be solved by separation of variables. If we assume a solution of the form

$$S=-K\vartheta-Et+p_zz+p_xx+W(y), \tag{50}$$

and introduce the new variable y',

$$y=y'+cp_x/eB, \tag{51}$$

Equation (47) becomes

$$\frac{A^2}{2m_0}=\frac{1}{2m_0}\left(\frac{dW}{dy'}\right)^2+\frac{e^2B^2}{2m_0c^2}y'^2, \tag{52}$$

where

$$A^2=2m_0K+\frac{E^2}{c^2}-p_z{}^2-m_0{}^2c^2. \tag{53}$$

Equation (52) is in the form of the energy equation for a one-dimensional harmonic oscillator, so that WKB quantization must lead to the eigenvalues

$$A^2/2m_0=(n+\tfrac{1}{2})\hbar\Omega, \tag{54}$$

where $\Omega=eB/m_0c$.[16] Equation (53) then becomes

$$K+\frac{1}{2m_0}\left(\frac{E^2}{c^2}-p_z{}^2-m_0{}^2c^2\right)=(n+\tfrac{1}{2})\hbar\Omega. \tag{55}$$

The solution for the spinor Φ is easily found to be

$$\Phi=\exp(-\tfrac{1}{2}i\Omega\sigma_z'\vartheta)\Phi_0, \tag{56}$$

where Φ_0 is a constant unit spinor.

The solution (46) is

$$\psi=\Phi_0(\tfrac{1}{2}A^2-\tfrac{1}{2}m_0{}^2\Omega^2y'^2)^{-\frac{1}{4}}\exp\left\{\frac{i}{\hbar}[(-K-\tfrac{1}{2}\hbar\Omega\sigma_z')\vartheta-Et+p_zz+p_xx+W(y)]\right\}. \tag{57}$$

[13] The proof of Eq. (41) is carried out in the Appendix.

[14] See reference 9.

[15] In Eq. (47), we cannot require that $\partial S/\partial\vartheta=0$, for we then neglect the spin energy contribution to the total energy of the charge.

[16] See, for example, the section on the WKB approximation in L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, New York, 1955), Sec. 28, 2nd ed.

Final quantization of this WKB theory is achieved by requiring that the wave function ψ satisfy the equation

$$i\hbar(\partial\psi/\partial\vartheta)=0. \tag{58}$$

The requirement (58) leads to the eigenvalue equation

$$(K+\tfrac{1}{2}\Omega\hbar\sigma_z')\Phi_0=0, \tag{59}$$

so that K takes the values $K=\frac{1}{2}\hbar\Omega\epsilon$, where $\epsilon=\pm 1$. The energy eigenvalues are then found from (53),

$$E=\pm[p_z^2c^2+m_0^2c^4+2Bec\hbar(\tfrac{1}{2}\epsilon+n+\tfrac{1}{2})]^{\frac{1}{2}}. \tag{60}$$

These are the energy levels predicted in the Dirac theory.[17]

The two independent positive energy solutions to the second-order Dirac equation are easily seen to be

$$\psi_+=R\exp\left[\frac{i}{\hbar}(S-\tfrac{1}{2}\Omega\sigma_z'\vartheta)\right]\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \tag{61a}$$

$$\psi_-=R\exp\left[\frac{i}{\hbar}(S-\tfrac{1}{2}\Omega\sigma_z'\vartheta)\right]\begin{pmatrix}0\\1\\0\\0\end{pmatrix}. \tag{61b}$$

VIII. KEPLER PROBLEM

The classical equations of motion (18′) for a spinning dipole in a Coulomb-field are

$$\frac{d\mathbf{P}}{d\vartheta}=-\frac{e^2V_4\mathbf{r}}{r^3}-\frac{e^2}{m_0c}\left[\frac{\mathbf{N}}{r^3}-\frac{3(\mathbf{N}\cdot\mathbf{r})\mathbf{r}}{r^5}\right], \tag{62a}$$

$$\frac{d\mathbf{M}}{d\vartheta}=-\frac{e^2}{m_0c}(\mathbf{N}\times\mathbf{r})\frac{1}{r^3}, \tag{62b}$$

$$\frac{d\mathbf{N}}{d\vartheta}=\frac{e^2}{m_0c}(\mathbf{M}\times\mathbf{r})\frac{1}{r^3}. \tag{62c}$$

These equations admit the following five constants of the motion:

$$\mathbf{J}=\mathbf{r}\times\mathbf{P}+\mathbf{M}, \tag{63a}$$

$$A=(\mathbf{r}\times\mathbf{P})^2+\frac{2e^2}{c}\frac{\mathbf{N}\cdot\mathbf{r}}{r}, \tag{63b}$$

$$E=c\left(P_r^2+\frac{A}{r^2}+m_0^2c^2\right)^{1/2}-\frac{e^2}{r}. \tag{63c}$$

For this problem, the Hamilton-Jacobi equation, (32), is

$$\left(\frac{\partial\mathcal{S}}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial\mathcal{S}}{\partial\theta}\right)^2+\frac{1}{r^2\sin^2\theta}\left(\frac{\partial\mathcal{S}}{\partial\varphi}\right)^2-\frac{1}{c^2}\left(E+\frac{e^2}{r}\right)^2+m_0^2c^2$$
$$-\frac{ie^2}{r^2c}\left[\left(\Sigma^2-4\left(\frac{\partial\mathcal{S}}{\partial\eta}\right)^2\right)^{1/2}\sin(\eta+\varphi)\sin\theta\right.$$
$$-\left(\Sigma^{*2}-4\left(\frac{\partial\mathcal{S}}{\partial\eta^*}\right)^2\right)^{1/2}\sin(\eta^*+\varphi)\sin\theta$$
$$\left.+2\left(\frac{\partial\mathcal{S}}{\partial\eta^*}-\frac{\partial\mathcal{S}}{\partial\eta}\right)\cos\theta\right]=0. \tag{64}$$

If Eq. (64) could be solved, we could then construct the spinor, (44), by the methods outlined in Sec. VI. The usual WKB conditions would then give the various eigenvalues for the problem. Unfortunately, we have been unable to find a complete solution to Eq. (64).

However, other techniques are available for solving Eq. (64), and although these methods do not yield the most general solutions, there is good reason to believe that, from these restricted solutions, one may derive in the WKB theory the same complete set of measurable eigenvalues as in the quantum theory.

In a previous paper[18] we gave the general form for quantum operators in the WKB approximation to the Schrödinger theory,

$$G_{\mathrm{op}}{}^{\mathrm{WKB}}=G\left(q,\frac{\partial S}{\partial q}\right)-\frac{i\hbar}{R}\frac{\partial R}{\partial q^i}\frac{\partial G}{\partial(\partial S/\partial q^i)}$$
$$-\frac{i\hbar}{2}\frac{\partial^2 G}{\partial q^i\partial(\partial S/\partial q^i)}. \tag{65}$$

These asymptotic operators satisfy eigenvalue equations as in the original quantum theory. For example, if one constructs the WKB wave function

$$\psi=R\exp(iS/\hbar),$$

for a problem with rotational symmetry, then this wave function satisfies the eigenvalue equation

$$L_{\mathrm{op}}{}^2\psi=\alpha_\theta^2\psi, \tag{66}$$

where α_θ^2 is the value of the square of the total orbital angular momentum, and $L_{\mathrm{op}}{}^2$ in Cartesian coordinates is

$$L_{\mathrm{op}}{}^2=(\mathbf{r}\times\nabla S)^2-\frac{2i\hbar}{R}\nabla R\cdot[\mathbf{r}(\mathbf{r}\cdot\nabla S)-(\nabla S)r^2]$$
$$-2i\hbar\mathbf{r}\cdot\nabla S. \tag{67}$$

The expectation values for the classical operator $L_{\mathrm{op}}{}^2$ are the classical relations for the square of the total angular momentum.

[17] M. H. Johnson and B. A. Lippmann, Phys. Rev. 76, 828 (1949).

[18] See reference 1, B.

In the spin theory, the form of the WKB operators is different from that given in Eq. (65), for the WKB spinor wave function is of the form

$$\psi = R\Phi \exp(iS/\hbar).$$

In addition, the momentum is defined as

$$P_\mu = \frac{\partial S}{\partial x^\mu} + \frac{1}{2}\left(\xi \frac{\partial \eta}{\partial x^\mu} + \xi^* \frac{\partial \eta^*}{\partial x^\mu}\right).$$

We shall not write the exact form of the WKB operators in the spin theory, but merely observe that they may be found by considering the effects to first order in $\hbar$ of the operators q_{op} and P_{op} acting on the WKB spinor, $\psi = R\Phi \exp(iS/\hbar)$.

We shall now find the asymptotic solutions for the Kepler problem in the Dirac theory by requiring that the WKB spinor wave functions satisfy the eigenvalue equation

$$[\mathfrak{L}_{\mathrm{op}}^2 - (ie^2\hbar/cr)(\boldsymbol{\alpha}\cdot\mathbf{r})]\psi = A\psi. \tag{68}$$

$\mathfrak{L}_{\mathrm{op}}^2$ is that WKB angular momentum operator of the spin theory for which $\mathfrak{L}_{\mathrm{op}}^2\psi_i = \alpha_{\theta_i}^2\psi_i$, and α_θ is the orbital angular momentum. Equation (68) is the WKB eigenvalue equation analogue of the classical relation (63b). In a representation in which the matrix γ_4 is diagonal, Eq. (68) becomes

$$\alpha_{\theta_1}^2\psi_1 - (i\hbar e^2/c)(\sin\theta\, e^{-i\varphi}\psi_4 + \cos\theta\,\psi_3) = A\psi_1, \tag{69a}$$

$$\alpha_{\theta_2}^2\psi_2 - (i\hbar e^2/c)(\sin\theta\, e^{i\varphi}\psi_3 - \cos\theta\,\psi_4) = A\psi_2, \tag{69b}$$

$$\alpha_{\theta_3}^2\psi_3 - (i\hbar e^2/c)(\sin\theta\, e^{-i\varphi}\psi_2 + \cos\theta\,\psi_1) = A\psi_3, \tag{69c}$$

$$\alpha_{\theta_4}^2\psi_4 - (i\hbar e^2/c)(\sin\theta\, e^{i\varphi}\psi_1 - \cos\theta\,\psi_2) = A\psi_4. \tag{69d}$$

We can find a solution to Eqs. (69) by separation of variables. Given the solutions

$$-ie^{-i\varphi}\psi_4 = \psi_3, \tag{70a}$$

$$ie^{-i\varphi}\psi_2 = \psi_1, \tag{70b}$$

$$-ie^{i\theta}\psi_3 = \beta\psi_1, \tag{70c}$$

$$ie^{i\theta}\psi_3 = \beta\psi_2, \tag{70d}$$

Equations (69) now appear as

$$\left(\alpha_{\theta_1}^2 + \frac{e^2\hbar}{c}\beta - A\right)\psi_1 = 0, \tag{71a}$$

$$\left(\alpha_{\theta_2}^2 + \frac{e^2\hbar}{c}\beta - A\right)\psi_2 = 0, \tag{71b}$$

$$\left(\alpha_{\theta_3}^2 - \frac{e^2\hbar}{c}\frac{1}{\beta} - A\right)\psi_3 = 0, \tag{71c}$$

$$\left(\alpha_{\theta_4}^2 - \frac{e^2\hbar}{c}\frac{1}{\beta} - A\right)\psi_4 = 0. \tag{71d}$$

We choose ψ_1 to be the function

$$\psi_1 = R(r)e^{i(m'-1)\varphi}\Theta_{l_1}{}^{m'}, \tag{72}$$

where $R(r)$ is a radial function to be determined, and $\Theta_{l_i}{}^{m'}$ is[19]

$$\Theta_{l_i}{}^{m'} = (\sin\theta)^{-\frac{1}{2}}(\alpha_{\theta_i}^2 - m'^2\hbar^2/\sin^2\theta)^{-1/4} \times \exp\left[i/\hbar \int (\alpha_{\theta_i}^2 - m'^2\hbar^2/\sin^2\theta)^{1/2} d\theta\right]. \tag{73}$$

m' is the integer associated with the z component of the orbital angular momentum. With this value for ψ_1, the relations (70), and the choice of constants $\alpha_{\theta_1} = \alpha_{\theta_2}$, $\alpha_{\theta_3} = \alpha_{\theta_4}$, we find that the solutions of Eq. (68) become

$$\psi_1 = R(r)e^{i(m'-1)\varphi}\Theta_{l_1}{}^{m'}, \tag{74a}$$

$$\psi_2 = iR(r)e^{im'\varphi}\Theta_{l_1}{}^{m'}, \tag{74b}$$

$$\psi_3 = i\beta R(r)e^{i(m'-1)\varphi}\Theta_{l_2}{}^{m'}e^{-i\theta}, \tag{74c}$$

$$\psi_4 = \beta R(r)e^{im'\varphi}\Theta_{l_2}{}^{m'}e^{-i\theta}. \tag{74d}$$

The α_θ's appearing in Eqs. (70) are the eigenvalues determined by the usual WKB requirement that each component of the spinor be single-valued. If we use the method of Keller[20] to secure these eigenvalues, we find that $\alpha_{\theta_1} = \alpha_{\theta_2} = (l+\frac{1}{2})\hbar$, $\alpha_{\theta_3} = \alpha_{\theta_4} = (l+\frac{3}{2})\hbar$. These values of α_θ may now be substituted back into the Eqs. (71) and the constant β is then determined,

$$\beta_{1,2} = (l+1)/\gamma \pm [(l+1)^2 - \gamma^2]^{1/2}/\gamma, \tag{75}$$

with $\gamma = e^2/\hbar c$.

From Eqs. (71), we find that A has the values

$$A_{1,2} = \{[(l+1)^2 - \gamma^2]^{1/2} \pm \tfrac{1}{2}\}^2\hbar^2 + \gamma^2\hbar^2. \tag{76}$$

We are now in a position to determine the energy eigenvalues. From Eq. (63) we see that the relativistic radial Hamilton-Jacobi equation is

$$\frac{1}{c^2}(E + e^2/r)^2 - (dS_r/dr)^2 - A_{1,2}/r^2 - m_0^2c^2 = 0. \tag{77}$$

The radial WKB wave function is easily found to be[21]

$$R(r)_{1,2} = r^{-1}[c^{-2}(E+e^2/r)^2 - (A_{1,2}/r^2) - m_0^2c^2]^{-1/4} \exp iP_r/\hbar. \tag{78}$$

The method of Keller[22] applied to this problem gives the exact Dirac eigenvalues

$$E_{1,2}/m_0c^2 = \{1 + \gamma^2/[k + \tfrac{1}{2} \mp \tfrac{1}{2} + ((l+1)^2 - \gamma^2)^{1/2}]^2\}^{-1/2}. \tag{79}$$

[19] $\Theta_l{}^{m'}$ is the eigenfunction of the operator L_{op}^2 of Eqs. (66)–(67).

[20] J. B. Keller, Ann. Phys. (New York) 4, 180 (1958); J. B. Keller and S. I. Rubinow, *ibid.* 9, 24 (1960).

[21] $R(r) = Cg^{-1/4}(\partial P_r/\partial E)^{1/2}V_4^{-1/2}\exp[(i/\hbar)\int P_r dr]$, where g is the radial dependent part of the determinant of the configuration space metric, g_{ik}; C is a constant, $P_r = dS_r/dr$, and $V_4 = (i/c)(E+e^2/r)$.

[22] See reference 20.

The eigenfunctions given by (73), (74), and (78) are the WKB approximation to the second-order Dirac equation. They are simultaneous eigenstates of the operators E, J_z, and A, as are the exact solutions of the second-order equation.[23]

IX. TRANSFORMATION PROPERTIES OF THE QUASI-CLASSICAL SPIN THEORY

The transformation properties of the quasi-classical spin theory are most easily deduced from the invariances of the Lagrangian, (26). It is clear that this Lagrangian is invariant under the following transformations:

(a) Lorentz transformations and spin rotations,

$$S^{-1}\gamma^\mu S = a_{\mu\nu}\gamma^\nu, \tag{80a}$$

$$x'^\mu = a_{\mu\nu}x^\nu, \tag{80b}$$

$$\psi' = S\psi, \quad \bar{\psi}' = \bar{\psi}S^{-1}, \tag{80c}$$

where the $a_{\mu\nu}$ are constants satisfying the orthogonality conditions

$$a_{\mu\rho}a_{\nu\rho} = a_{\rho\mu}a_{\rho\nu} = \delta_{\mu\nu}; \tag{81}$$

(b) Gauge transformation of the second kind,

$$A_\mu' = A_\mu + \partial\Lambda/\partial x^\mu), \quad \psi' = \psi \exp(ie\Lambda/\hbar c); \tag{82}$$

(c) Charge conjugation,

$$\psi' \equiv \psi_{\text{c.c.}} = \gamma_2\psi^*, \quad e \to -e. \tag{83}$$

In addition, we have the general canonical transformations, which leave invariant the form of Hamilton's equations, as well as the form of the fundamental spinors ψ of (25) or (44). On the other hand, these transformations change the form of the Hamiltonian operators of (27) or (45). The canonical transformations are generated by a function $\mathfrak{F} = \mathfrak{F}(x_\mu, X_\mu, \eta, \mathrm{H})$,

$$\mathcal{S}'(X^\mu, \mathrm{H}, \alpha^A) = \mathcal{S}(x^\mu, \eta, \alpha^A) + \mathfrak{F}, \tag{84}$$

where the x^μ, η, and η^* are the original configuration space variables, and the X^μ, H, and H* the new configuration space variables.

Given the transformed Hamilton-Jacobi function $\mathcal{S}'$, we can then find new variables Ξ and ρ' with which to construct a spinor similar in form to ψ of (25) or (44). The detailed construction of these new variables is completely analogous to the corresponding solution for the nonrelativistic canonical spin theory of C, Sec. VIII,[1] and will therefore not be given in this paper.

X. CONCLUSION

We have presented a different version of the WKB theory for relativistic charged particles with internal degrees of freedom. Our starting point has been the classical Hamilton-Jacobi theory for a spinning particle, and we have shown how the solutions to this classical equation could be used to construct a relativistic spinor which satisfies an equation similar to the second-order Dirac equation. We have quantized the classical theory by requiring that our spinor wave functions be continuous and single valued, and have found that for a particle in a magnetic field, and again in a Coulomb field, our theory leads to the energy eigenvalues of the Dirac theory.

Our solutions are asymptotic approximations to the second-order Dirac equation. These solutions can be converted into solutions of an "asymptotic" first-order Dirac equation by several different schemes. One of these schemes seems to be preferred to others, in that, for some problems in special coordinate systems, it provides us with exact solutions of the first order Dirac equation. We have not incorporated this material in the present paper, because, as yet, we do not understand its dynamical significance. We hope to discuss this problem before long in another paper.

Beyond the question of the dynamical significance of a solution to the first-order Dirac equation, there is the problem of finding asymptotic solutions for a mass- and charge-renormalized electron theory. We believe that such solutions may be found by extending the framework of the WKB scheme presented in the present paper.

APPENDIX

We show that the Hamilton-Jacobi equation, (39), gives rise to the equation of continuity, (41).

Assume that the dynamical equations are derivable from a Hamiltonian H,

$$H = H(x^\mu, P_\mu, \xi, \xi^*, \eta, \eta^*) = 0, \tag{A1}$$

where

$$P_\mu = \frac{\partial S}{\partial x^\mu} + \tfrac{1}{2}\xi\frac{\partial \eta}{\partial x^\mu} + \tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial x^\mu}, \tag{A2}$$

and the canonical spin equations are

$$\frac{1}{2}\frac{\partial \xi}{\partial x^\mu}V^\mu = \frac{\partial H}{\partial \eta}, \quad \frac{1}{2}\frac{\partial \eta}{\partial x^\mu}V^\mu = -\frac{\partial H}{\partial \xi}; \tag{A3a}$$

$$\frac{1}{2}\frac{\partial \xi^*}{\partial x^\mu}V^\mu = \frac{\partial H}{\partial \eta^*}, \quad \frac{1}{2}\frac{\partial \eta^*}{\partial x^\mu}V^\mu = -\frac{\partial H}{\partial \xi^*}. \tag{A3b}$$

The four-velocity V^μ is defined through the canonical equation

$$V^\mu = \partial H/\partial P_\mu. \tag{A4}$$

We differentiate H with respect to the three constants α_i and find

$$\frac{\partial H}{\partial \alpha_i} = \frac{\partial H}{\partial P_\mu}\frac{\partial P_\mu}{\partial \alpha_i} + \frac{\partial H}{\partial(\eta/2)}\frac{\partial(\eta/2)}{\partial \alpha_i} + \frac{\partial H}{\partial \xi}\frac{\partial \xi}{\partial \alpha_i} + \text{c.c.} = 0. \tag{A5}$$

[23] Compare our results with the solutions to the squared Dirac equation given in A. Sokolow, *Quantenelektrodynamik* (Akademie-Verlag, Berlin, 1957), pp. 293–296.

If we now insert (A2) in (A5), we have after a short calculation

$$\frac{\partial H}{\partial P_\mu}\left\{\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_i}+\tfrac{1}{2}\xi\frac{\partial \eta}{\partial \alpha_i}+\tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial \alpha_i}\right)\right\}+\frac{\partial \xi}{\partial \alpha_i}\left(\frac{\partial H}{\partial P_\mu}\frac{\partial(\eta/2)}{\partial x^\mu}+\frac{\partial H}{\partial \xi}\right)+\frac{\partial(\eta/2)}{\partial \alpha_i}\left(\frac{\partial H}{\partial(\eta/2)}-\frac{\partial H}{\partial P_\mu}\frac{\partial \xi}{\partial x^\mu}\right)+\text{c.c.}=0. \quad \text{(A6)}$$

The coefficients of $\partial\xi/\partial\alpha_i$ and $\partial(\eta/2)/\partial\alpha_i$ vanish because of (A3). We expand the remainder so that (A6) reads

$$(ic)^{-1}\frac{\partial H}{\partial P_4}\frac{\partial \beta_i}{\partial t}+\frac{\partial H}{\partial P_k}\frac{\partial \beta_i}{\partial x_k}=0, \quad \text{(A7)}$$

where

$$\beta_i=\frac{\partial S}{\partial \alpha_i}+\tfrac{1}{2}\xi\frac{\partial \eta}{\partial \alpha_i}+\tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial \alpha_i}.$$

We divide (A7) by $(ic)^{-1}(\partial H/\partial P_4)$, and then differentiate with respect to x^j. The resulting equation is

$$\frac{\partial}{\partial t}\frac{\partial \beta_i}{\partial x^j}+\frac{\partial v^k}{\partial x^j}\frac{\partial \beta_i}{\partial x^k}+v^k\frac{\partial^2 \beta_i}{\partial x^k \partial x^j}=0, \quad \text{(A8)}$$

with

$$v^k\equiv\frac{dx^k}{dt}=ic\frac{\partial H}{\partial P_k}\Big/\frac{\partial H}{\partial P_4}.$$

We define the matrix $\phi_{ij}=\partial\beta_i/\partial x^j$, its inverse ϕ^{ij},

$$\phi^{im}\phi_{ni}=\phi^{mi}\phi_{in}=\delta_n{}^m,$$

and the determinant $\|\phi_{ij}\|=\bar{\rho}$.

We multiply (A8) by ϕ^{ij} and find the desired result, (41),

$$\partial\bar{\rho}/\partial t+(\partial/\partial x^k)(\bar{\rho}v^k)=0. \quad \text{(A9)}$$

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Description of Particles in Quantized Field Theory*

P. J. Peebles
Palmer Physical Laboratory, Princeton, New Jersey
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The problem of establishing the connection between quantized field theory and observations is discussed in a nonrigorous way without recourse to the idea of asymptotic free fields. The method is based on a prescription for the state vector associated with a single physical particle using the interacting field operator. It is shown that the method leads to the usual form for the scattering matrix, obtained via the asymptotic condition. As a second application, the problem of describing an unstable particle is discussed briefly.

1. INTRODUCTION

The asymptotic condition[1] has played an important role in establishing the connection between quantized field theory and observations. This condition gives expression to the idea that any physical system, taken in the sufficiently remote past or future, must consist of a collection of stable particles which are so far apart that they do not interact with each other. It should be possible to describe this situation with a free-field theory. Thus, it is argued that in any reasonable field theory matrix elements of the field operators, for arguments in the very remote past, should be "equal" to the corresponding matrix elements of free operators.

Evidently, if the conventional, many-field Lagrangian theories such as quantum electrodynamics are physically reasonable, it should be possible to understand how the asymptotic condition follows from the basic principles of the theory. The purpose of this article is to discuss some aspects of this question and, in particular, to give a simple, nonrigorous description of a scattering experiment without recourse to the idea of asymptotic free fields. It should be remarked that we do not prove that the field theory satisfies the asymptotic condition. Rather, we describe a method for establishing the physical significance of the quantized field theory, without explicit reference to asymptotic fields. This leads to a formula for the scattering matrix, and, assuming that this formula is well defined, it is seen to be the usual form for the scattering matrix element.

Related to the above discussion is the problem of finding a treatment of unstable elementary particles in a many-field theory. In the asymptotic field approach, an unstable particle would be constructed as a resonance in the interaction of stable incident particles. A different approach is discussed briefly. It is shown that the condition that the particle should decay according to

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[1] H. Lehmann, K. Symanzik, and W. Zimmerman, Nuovo cimento 1, 205 (1955).