If we now insert (A2) in (A5), we have after a short calculation

$$\frac{\partial H}{\partial P_\mu}\left\{\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial \alpha_i}+\tfrac{1}{2}\xi\frac{\partial \eta}{\partial \alpha_i}+\tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial \alpha_i}\right)\right\} +\frac{\partial \xi}{\partial \alpha_i}\left(\frac{\partial H}{\partial P_\mu}\frac{\partial(\eta/2)}{\partial x^\mu}+\frac{\partial H}{\partial \xi}\right) +\frac{\partial(\eta/2)}{\partial \alpha_i}\left(\frac{\partial H}{\partial(\eta/2)}-\frac{\partial H}{\partial P_\mu}\frac{\partial \xi}{\partial x^\mu}\right)+\text{c.c.}=0. \quad \text{(A6)}$$

The coefficients of $\partial\xi/\partial\alpha_i$ and $\partial(\eta/2)/\partial\alpha_i$ vanish because of (A3). We expand the remainder so that (A6) reads

$$(ic)^{-1}\frac{\partial H}{\partial P_4}\frac{\partial \beta_i}{\partial t}+\frac{\partial H}{\partial P_k}\frac{\partial \beta_i}{\partial x_k}=0, \quad \text{(A7)}$$

where

$$\beta_i=\frac{\partial S}{\partial \alpha_i}+\tfrac{1}{2}\xi\frac{\partial \eta}{\partial \alpha_i}+\tfrac{1}{2}\xi^*\frac{\partial \eta^*}{\partial \alpha_i}.$$

We divide (A7) by $(ic)^{-1}(\partial H/\partial P_4)$, and then differentiate with respect to x^j. The resulting equation is

$$\frac{\partial}{\partial t}\frac{\partial \beta_i}{\partial x^j}+\frac{\partial v^k}{\partial x^j}\frac{\partial \beta_i}{\partial x^k}+v^k\frac{\partial^2\beta_i}{\partial x^k\partial x^j}=0, \quad \text{(A8)}$$

with

$$v^k\equiv\frac{dx^k}{dt}=ic\frac{\partial H}{\partial P_k}\Big/\frac{\partial H}{\partial P_4}.$$

We define the matrix $\phi_{ij}=\partial\beta_i/\partial x^j$, its inverse ϕ^{ij},

$$\phi^{im}\phi_{ni}=\phi^{mi}\phi_{in}=\delta_n{}^m,$$

and the determinant $\|\phi_{ij}\|=\bar{\rho}$.

We multiply (A8) by ϕ^{ij} and find the desired result, (41),

$$\partial\bar{\rho}/\partial t+(\partial/\partial x^k)(\bar{\rho}v^k)=0. \quad \text{(A9)}$$

PHYSICAL REVIEW VOLUME 128, NUMBER 3 NOVEMBER 1, 1962

Description of Particles in Quantized Field Theory*

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(Received May 17, 1962)

The problem of establishing the connection between quantized field theory and observations is discussed in a nonrigorous way without recourse to the idea of asymptotic free fields. The method is based on a prescription for the state vector associated with a single physical particle using the interacting field operator. It is shown that the method leads to the usual form for the scattering matrix, obtained via the asymptotic condition. As a second application, the problem of describing an unstable particle is discussed briefly.

1. INTRODUCTION

THE asymptotic condition[1] has played an important role in establishing the connection between quantized field theory and observations. This condition gives expression to the idea that any physical system, taken in the sufficiently remote past or future, must consist of a collection of stable particles which are so far apart that they do not interact with each other. It should be possible to describe this situation with a free-field theory. Thus, it is argued that in any reasonable field theory matrix elements of the field operators, for arguments in the very remote past, should be "equal" to the corresponding matrix elements of free operators.

Evidently, if the conventional, many-field Lagrangian theories such as quantum electrodynamics are physically reasonable, it should be possible to understand how the asymptotic condition follows from the basic principles of the theory. The purpose of this article is to discuss some aspects of this question and, in particular, to give a simple, nonrigorous description of a scattering experiment without recourse to the idea of asymptotic free fields. It should be remarked that we do not prove that the field theory satisfies the asymptotic condition. Rather, we describe a method for establishing the physical significance of the quantized field theory, without explicit reference to asymptotic fields. This leads to a formula for the scattering matrix, and, assuming that this formula is well defined, it is seen to be the usual form for the scattering matrix element.

Related to the above discussion is the problem of finding a treatment of unstable elementary particles in a many-field theory. In the asymptotic field approach, an unstable particle would be constructed as a resonance in the interaction of stable incident particles. A different approach is discussed briefly. It is shown that the condition that the particle should decay according to

* Supported in part by research contracts with the U. S. Atomic Energy Commission and the Office of Naval Research.

[1] H. Lehmann, K. Symanzik, and W. Zimmerman, Nuovo cimento 1, 205 (1955).

the usual decay law leads to some results which have been obtained previously by other arguments.

The assumptions which will be used are listed in Sec. 2. All of the assumptions are familiar and have been used by many authors. In Sec. 3 the state vector for a single particle is prescribed. This is taken to consist of an appropriate linear function of the interacting field operator, operating on the vacuum state. The state vector for a localized particle is constructed and is shown that this system moves with time according to the elementary properties of a wave packet for a single particle. In Sec. 4 these results are used to construct the scattering matrix. In Sec. 5 the methods are applied to a discussion of unstable particles.

2. DESCRIPTION OF THE THEORY

The field theory to be considered is taken to have the following properties. For simplicity, we shall suppose that there is a single real scalar field operator, $\phi(x)$, with one stable particle.

The equations are written for unit quantization volume, with Cartesian coordinates, where the fourth component of a vector is imaginary.

A. Covariance

The theory is covariant against Lorentz transformations of the coordinates used to describe a system. The energy and momentum operators satisfy the equations

$$[P_\nu,\phi(x)]=i\partial\phi(x)/\partial x^\nu,$$
$$[P_\nu,P_\sigma]=0, \tag{1}$$

with indices running from one to four.

B. Field Equation and Commutation Relations

The field operator satisfies a field equation

$$(\mu^2-\Box)\phi(x)=J(x), \tag{2}$$

and the commutation relation

$$[\phi(x),\phi(y)]=0, \tag{3}$$

for points x and y which are separated by a space-like interval. In the field equation, $J(x)$ is a real scalar field operator constructed from $\phi(x)$, and μ is the physical mass of the stable particle in the theory.

C. Complete Set of States

There exists a vacuum state, which has zero energy and zero momentum. It is a member of a complete set of orthonormal states with sharp energy and momentum,

$$P_\nu|n,\alpha\rangle=k_\nu{}^n|n,\alpha\rangle, \tag{4}$$

where α is supposed to include the additional quantum numbers necessary to specify the state.

D. The Spectral Representation

The vacuum expectation value of the product of two field operators satisfies the representation[2]

$$\langle 0|\phi(x)\phi(y)|0\rangle=\int\rho(m^2)dm^2\Delta^{(+)}(x-y;m^2), \tag{5}$$

where $\Delta^{(+)}(x-y;m^2)$ is the singular invariant function corresponding to a free particle of mass m, and

$$\rho(-k^{n2})=\textstyle\sum_\alpha |\langle n,\alpha|\phi(0)|0\rangle|^2. \tag{6}$$

Here the sum is taken over all states with momentum $k_\nu{}^n$. It is assumed that the field operator connects the vacuum state with physical states only, that is, with states with positive energy and mass equal to μ or greater than or equal to 2μ. The field operator is normalized so that

$$\rho(-k^2)=\delta(k^2+\mu^2)+\sigma(-k^2), \tag{7}$$

where $\sigma(-k^2)$ vanishes for $-k^2<4\mu^2$.

3. THE STATE VECTOR FOR A SINGLE PARTICLE

It is customary to define the time-dependent operators

$$a_k(x^0)=i\int d^3x\frac{e^{-ik\cdot x}}{(2k_0)^{1/2}}\frac{\overleftrightarrow{\partial}}{\partial x^0}\phi(x), \tag{8}$$

where

$$k_0=(\mathbf{k}^2+\mu^2)^{1/2}, \tag{9}$$

and as usual

$$A\left(\frac{\overleftrightarrow{\partial}}{\partial x^0}\right)B\equiv A\frac{\partial B}{\partial x^0}-B\frac{\partial A}{\partial x^0}.$$

For a theory without interactions, where the right-hand side of the field equation (2) vanishes, the customary procedure is to remark that it is consistent and necessary to assume that the vacuum state satisfies

$$a_k|0\rangle=0 \tag{10}$$

(for otherwise this state would have negative energy), and assuming the commutation relation (3), and the second canonical commutation relation

$$[\phi(\mathbf{x},t),\,\partial\phi(\mathbf{x}',t)/\partial t]=i\delta(\mathbf{x}-\mathbf{x}'), \tag{11}$$

it may be concluded that the operators $a_k{}^\dagger$, a_k are creation and annihilation operators for free particles with momentum equal to $\mathbf{k}$ and mass equal to μ.

For a theory with interactions it is known that this procedure is not adequate. However, it is still of interest to consider the two state vectors

$$a_k{}^\dagger(x^0)|0\rangle, \tag{12}$$

$$a_k(x^0)|0\rangle, \tag{13}$$

where the time x^0 appearing in these states is to be

[2] H. Lehmann, Nuovo cimento 11, 342 (1954).

regarded as a parameter. It is immediately apparent from Eq. (1) and the definition (8) that the states have momenta equal to $\mathbf{k}$ and $-\mathbf{k}$, respectively. The energy spectra of the states follow from the matrix elements with the complete set (4),

$$\langle n,\alpha|a_k^\dagger(x^0)|0\rangle =\frac{k_0{}^n+k_0}{(2k_0)^{1/2}}\langle n,\alpha|\phi(0)|0\rangle e^{i(k_0{}^n-k_0)x0}\delta(\mathbf{k}^n-\mathbf{k}), \tag{14}$$

$$\langle n,\alpha|a_k(x^0)|0\rangle =\frac{k_0{}^n-k_0}{(2k_0)^{1/2}}\langle n,\alpha|\phi(0)|0\rangle e^{i(k_0{}^n+k_0)x0}\delta(\mathbf{k}^n-\mathbf{k}). \tag{15}$$

It will be recalled that $k_0=(\mathbf{k}^2+\mu^2)^{1/2}$, while $k_0{}^n$ is the energy of the state $|n,\alpha\rangle$. We have used (8) and the formula

$$\langle n,\alpha|\phi(x)|0\rangle=e^{-ik^n\cdot x}\langle n,\alpha|\phi(0)|0\rangle, \tag{16}$$

which follows directly from Eqs. (1) and (4).

To construct a state vector which describes a single particle of mass μ, it is only necessary to remove the higher energy part of the state (12). This is done in the following way. Let

$$|k,x^0\rangle\equiv\int_{x^0-\Delta}^{x^0+\Delta}\frac{dx^{0\prime}}{2\Delta}a_k^\dagger(x^{0\prime})|0\rangle, \tag{17}$$

where Δ is a fixed parameter. By Eq. (14), this state vector satisfies the equations

$$\langle n,\alpha|k,x^0\rangle=\frac{k_0{}^n+k_0}{(2k_0)^{1/2}}\langle n,\alpha|\phi(0)|0\rangle\delta(\mathbf{k}^n-\mathbf{k}) \times\frac{\sin(k_0{}^n-k_0)\Delta}{(k_0{}^n-k_0)\Delta}e^{i(k_0{}^n-k_0)x0}. \tag{18}$$

Thus, using Eq. (7), the energy and momentum spectrum of (17) is

$$\sum_\alpha|\langle n,\alpha|k,x^0\rangle|^2=\frac{(k_0{}^n+k_0)^2}{2k_0}[\delta(k^{n2}+\mu^2)+\sigma(-k^{n2})] \times\delta(\mathbf{k}^n-\mathbf{k})\left[\frac{\sin(k_0{}^n-k_0)\Delta}{(k_0{}^n-k_0)\Delta}\right]^2. \tag{19}$$

Note that $\sigma(-k^{n^2})$ vanishes if $-k^{n^2}<4\mu^2$. Thus, if Δ is chosen such that

$$\mu^2/k_0\gg1/\Delta, \tag{20}$$

Eq. (19) reduces to the approximate form

$$\textstyle\sum_\alpha|\langle n,\alpha|k,x^0\rangle|^2\cong2k_0\delta(k^{n^2}+\mu^2)\delta(\mathbf{k}^n-\mathbf{k}) \cong\delta(k_0{}^n-k_0)\delta(\mathbf{k}^n-\mathbf{k}). \tag{21}$$

That is, by taking a linear superposition of the states (12) over a sufficiently large range Δ of the parameter x^0, we have been able to construct a state vector which almost certainly will be found to consist of a single particle. Also, it may be verified by Eq. (15) that, for sufficiently large Δ,

$$\int_{x^0-\Delta}^{x^0+\Delta}\frac{dx^{0\prime}}{2\Delta}a_k(x^{0\prime})|0\rangle\cong0. \tag{22}$$

To describe scattering processes, it is necessary to construct the state vector for a single, localized particle. Consider[3]

$$\bar{a}_x{}^\dagger|0\rangle, \tag{23}$$

where

$$\bar{a}_x\equiv\sum_{\mathbf{k},\mathbf{k}^2<m^2}e^{ik\cdot x}\int_{x^0-\Delta}^{x^0+\Delta}\frac{dx^{0\prime}}{2\Delta}a_k(x^{0\prime}). \tag{24}$$

As usual, $k_0=(\mathbf{k}^2+\mu^2)^{1/2}$. The sum is taken over all three-momenta $\mathbf{k}$ such that $\mathbf{k}^2<m^2$, where m is some large, fixed mass. Then using (20), we can choose the interval Δ large enough that the mass of the state (23) almost certainly is equal to μ. This state consists of a single particle which at time x^0 is to be found near the point $\mathbf{x}$. To justify this, we shall show that

$$[\bar{a}_x,\bar{a}_{x'}]=0,$$
$$[\bar{a}_x,\bar{a}_{x'}{}^\dagger]=0, \tag{25}$$

if x and x' are separated by a sufficiently large space-like distance. These commutation relations, with Eq. (22), imply that (23) should be interpreted as a system which is localized near the point $\mathbf{x}$ at time x^0.

By the definitions (8), (9), and (24),

$$[\bar{a}_x,\bar{a}_{x'}]=\sum_{\mathbf{k},\mathbf{k}'(<m^2)}e^{ik\cdot x}e^{ik'\cdot x'}\int_{x^0-\Delta}^{x^0+\Delta}\frac{d^4y}{2\Delta}\int_{x^{0\prime}-\Delta}^{x^{0\prime}+\Delta}\frac{d^4y'}{2\Delta} \times\left[\frac{e^{-ik\cdot y}}{(2k_0)^{1/2}}\frac{\overleftrightarrow{\partial}}{\partial y^0}[\phi(y),\phi(y')]\frac{\overleftrightarrow{\partial}}{\partial y^{0\prime}}\frac{e^{-ik'\cdot y'}}{(2k_0')^{1/2}}\right]. \tag{26}$$

Consider first the integral over $\mathbf{k}$. This vanishes if the point $(\mathbf{y},y_0)$ is well removed (by a distance large compared with the Compton wavelength of a particle) from the light cone of $(\mathbf{x},x^0)$. Since the difference between x^0 and y^0 is less than Δ, the integral vanishes if $|\mathbf{y}-\mathbf{x}|$ is much greater than Δ. A similar remark follows for $|\mathbf{y}'-\mathbf{x}'|$. Thus, if the point $(\mathrm{x}',x^{0\prime})$ is well removed (by a distance sufficiently large compared to Δ) from the light cone of $(\mathbf{x},x^0)$, the commutator in the right-hand side of (26) need be evaluated only for space-like separations of the arguments y and y'. By the locality condition [Eq. (3)], this commutator vanishes.

Clearly, the same argument may be applied to the second of the commutators (25).

[3] In the free-field theory, neglecting the mass cutoff, the state (23) would coincide with the definition of a single localized particle which was given by Newton and Wigner [Revs. Modern Phys. **21**, 400 (1949)].

4. DESCRIPTION OF A SCATTERING PROCESS

The state (23) consists of a single particle which at time x^0 is in a region of space with dimensions of the order of Δ. Evidently, it is possible to construct in the same way a system which at a given time consists of a definite number of localized particles, if we are willing to place the particles sufficiently far apart. This situation is suitable for discussions of scattering experiments. The case of nonforward elastic scattering will be considered here.

Following the definition (23), the state vector for a single particle which at the time x^0 is near the point $\mathbf{x}$, and which has momentum roughly equal to $\mathbf{k}$, is

$$\bar{a}_{k,x}{}^{\dagger}|0\rangle, \tag{27}$$

where

$$\bar{a}_{k,x}\equiv\sum_{\mathbf{p}}\omega(\mathbf{p}-\mathbf{k})e^{ip\cdot x}\int_{x^0-\Delta}^{x^0+\Delta}\frac{dy^0}{2\Delta}a_p(y^0), \tag{28}$$

and where the operator on the right-hand side was defined by Eqs. (8) and (9). The normalization of the real, non-negative weight function ω is taken to be

$$\sum\nolimits_{\mathbf{p}}\omega^2(\mathbf{p}-\mathbf{k})=1, \tag{29}$$

so that the state vector (27) is normalized.

To find the motion with time of the system (27), it is necessary to evaluate the transition amplitude

$$\langle 0|\bar{a}_{k,x}\bar{a}_{k'x'}{}^{\dagger}|0\rangle. \tag{30}$$

But we have the approximate equation

$$\int_{x^0-\Delta}^{x^0+\Delta}\frac{dy^0}{2\Delta}\int_{x^{0\prime}-\Delta}^{x^{0\prime}+\Delta}\frac{dy^{0\prime}}{2\Delta}\langle 0|a_p(y^0)a_{p'}{}^{\dagger}(y^{0\prime})|0\rangle=\delta(\mathbf{p}-\mathbf{p}'), \tag{31}$$

which is accurate if Δ satisfies the condition (20). This is obtained by inserting a sum over the complete set of states (4) between the operators in the matrix element in Eq. (31), and using the definition (8) and Eqs. (5), (6), and (7). The matrix element (30) is evaluated using Eqs. (28) and (31), and following the usual method one readily verifies that the result is consistent with the motion of the wave packet of a single free particle. In particular, the system (27) moves with velocity roughly equal to $\mathbf{k}/k_0$, and the system remains normalized to one particle.

The normalized state vector for two particles would be

$$\bar{a}_{k_1x_1}{}^{\dagger}\bar{a}_{k_2x_2}{}^{\dagger}|0\rangle, \tag{32}$$

where the points x_1 and x_2 are separated by a large space-like distance. For a scattering process, we are interested in the transition amplitude,

$$\bar{S}=\langle 0|\bar{a}_{k_1x_1}\bar{a}_{k_2x_2}\bar{a}_{k_3x_3}{}^{\dagger}\bar{a}_{k_4x_4}{}^{\dagger}|0\rangle, \tag{33}$$

between two particle states. The scattering matrix is obtained in the limit where $x_1{}^0=x_2{}^0=t_0$ is a time in the very remote future, $x_3{}^0=x_4{}^0=t_i$ is a time in the remote past, and where the momenta of the particles are well defined.

It will be convenient to use the definition

$$a_{k,x}(y^0)=\sum\nolimits_{\mathbf{p}}\omega(\mathbf{p}-\mathbf{k})e^{ip\cdot x}a_p|(y^0), \tag{34}$$

so that, by Eq. (28),

$$\bar{a}_{k,x}=\int_{x^0-\Delta}^{x^0+\Delta}\frac{dy^0}{2\Delta}a_{k,x}(y^0), \tag{35}$$

and Eq. (33) becomes

$$\bar{S}=\int_{t_0-\Delta}^{t_0+\Delta}\frac{dy_1{}^0}{2\Delta}\int_{t_0-\Delta}^{t_0+\Delta}\frac{dy_2{}^0}{2\Delta}\int_{t_i-\Delta}^{t_i+\Delta}\frac{dy_3{}^0}{2\Delta}\int_{t_i-\Delta}^{t_i+\Delta}\frac{dy_4{}^0}{2\Delta}S, \tag{36}$$

where

$$S=\langle 0|a_{k_1,x_1}(y_1{}^0)a_{k_2,x_2}(y_2{}^0)a_{k_3,x_3}{}^{\dagger}(y_3{}^0)a_{k_4,x_4}{}^{\dagger}(y_4{}^0)|0\rangle. \tag{37}$$

We have set $x_1{}^0=x_2{}^0=t_0$ (a time in the remote future), and $x_3{}^0=x_4{}^0=t_i$ (a time in the remote past).

The matrix element (37) may be simplified by applying a reduction formula. It follows from Eqs. (2) and (8) that

$$\frac{da_k{}^{\dagger}(z^0)}{dz^0}=-i\int d^3z\, e^{ik\cdot z}J(z)(2k_0)^{-1/2}. \tag{38}$$

Using Eqs. (34) and (38), the matrix element (37) may be written in the form

$$S=\langle 0|a_{k_1,x_1}(y_1{}^0)a_{k_2,x_2}(y_2{}^0)a_{k_3,x_3}{}^{\dagger}(t_0)a_{k_4,x_4}{}^{\dagger}(y_4{}^0)|0\rangle$$
$$+i\sum_{\mathbf{p}}\omega_3(\mathbf{p}-\mathbf{k}_3)e^{-ip\cdot x_3}\int_{y_3{}^0}^{t_0}d^4z\, H(z)e^{ip\cdot z}(2p_0)^{-1/2}, \tag{39}$$

where

$$H(z)=\langle 0|a_{k_1,x_1}(y_1{}^0)a_{k_2,x_2}(y_2{}^0)J(z)a_{k_4,x_4}{}^{\dagger}(y_4{}^0)|0\rangle. \tag{40}$$

To reduce this result to the usual form for the scattering matrix, it is necessary to show that terms such as the first one on the right-hand side of (39), which arise every time we contract a particle, vanish for nonforward scattering. It will be sufficient to show that, for example,

$$[a_{k_1,x_1}(y_1{}^0),\, a_{k_3,x_3}{}^{\dagger}(t_0)]=0, \tag{41}$$

where it will be recalled that particle one is in the scattered packet and particle three in the initial packet, and that $|y_1{}^0-t_0|\leq\Delta$. That is, we will show that it is possible to arrange the operators in the matrix element so that by Eqs. (22), (34), and (36) the matrix element does not affect the value of the scattering matrix $\bar{S}$.

The proof of Eq. (41) is almost identical with the discussion of Eq. (26). In detail, we have, by Eqs. (8) and (34),

$$[a_{k_1,x_1}(y_1{}^0),\, a_{k_3,x_3}{}^\dagger(t_0)]$$
$$=-\sum_{\mathbf{p}_1\mathbf{p}_3}\omega(\mathbf{p}_1-\mathbf{k}_1)\omega(\mathbf{p}_3-\mathbf{k}_3)e^{i p_1\cdot x_1}e^{-i p_3\cdot x_3}$$
$$\times\left(\frac{e^{-ip_1\cdot y_1}}{(2p_{10})^{1/2}}\frac{\overleftrightarrow{\partial}}{2y_1{}^0}[\phi(y_1),\phi(y_3)]\frac{\overleftrightarrow{\partial}}{\partial y_3{}^0}\frac{e^{ip_3\cdot y_3}}{(2p_{30})^{1/2}}\right), \quad (42)$$

where here $y_3{}^0=t_0$. Consider first the integral over $\mathbf{p}_3$. This vanishes unless

$$\mathbf{y}_3\cong\mathbf{x}_3+(\mathbf{k}_3/k_{30})(t_0-x_3{}^0). \quad (43)$$

Similarly, the term vanishes unless

$$|\mathbf{x}_1-\mathbf{y}_1|\cong 0. \quad (44)$$

But for nonforward scattering, the point $\mathbf{x}_3+(t_0-x_3{}^0)\times\mathbf{k}_3k_{30}{}^{-1}$ is well removed from the point $\mathbf{x}_1$. That is, the commutator in the right-hand side of (42) need be evaluated only for space-like separations of y_1 and y_3, and by the causality condition (3) this commutator vanishes. That is, we have found the commutation relation (41).

On repeated application of the above contraction method, the matrix S [Eq. (37)] may be reduced finally to the form

$$S=(i)^4\int_{t_i}^{y_1{}^0}d^4z_1\cdots\int_{y_4{}^0}^{t_0}d^4z_4\frac{e^{-ip_1\cdot z_1}}{(2p_{10})^{1/2}}\cdots\frac{e^{ip_3\cdot z_3}}{(2p_{40})^{1/2}}$$
$$\times(\mu^2-\square_1)\cdots(\mu^2-\square_4)$$
$$\times\langle 0|T(\phi(z_1)\cdots\phi(z_4))|0\rangle, \quad (45)$$

where we have not written here the sums over momenta $\mathbf{p}_i$ which give wave packets, or the matrix elements such as the first term on the right-hand side of Eq. (39), which vanish when S is substituted into Eq. (36).

Now, we assume that Eq. (45) is well defined as the limits of integration go to infinity. In this case, (45) must be independent of the choice of limits of integration if these limits are sufficiently far into the remote past and future. Then by Eq. (36), $\bar{S}=S$. That is, we see by Eq. (45) that the scattering matrix $\bar{S}$ has the limiting form

$$\bar{S}=i^4\int_{-\infty}^{\infty}d^4z_1\cdots d^4z_4\frac{e^{-ip_1\cdot z_1}}{(2p_{10})^{1/2}}\cdots\frac{e^{ip_4\cdot z_4}}{(2p_{40})^{1/2}}$$
$$\times(\mu^2-\square_1)\cdots(\mu^2-\square_4)$$
$$\times\langle 0|T(\phi(z_1)\cdots\phi(z_4))|0\rangle, \quad (46)$$

in the case where the momenta of the particles become sharply defined. This is the conventional form which is derived with the help of the asymptotic condition.[1]

In this section we have described a wave-packet picture of scattering processes. The state vector for the incident, well-separated particles was constructed using the interacting field operator, rather than an asymptotic free field. The method was to introduce a linear superposition of the vector (12), with the parameter x^0 within a large but finite range Δ. We could conclude, with arbitrarily good certainty, that this state consists of a single particle, if we took Δ sufficiently large. We used this result to construct the state vector for a collection of well-separated particles, each of which is localized to a region of space with dimensions of the order of Δ. With these results, it was immediately apparent how to write down the transition amplitude for a scattering process. Nonforward, elastic scattering was considered and the conventional form for the scattering matrix was obtained. It is concluded that the discussion of the physical significance of quantized field theories which has been presented here is equivalent to the usual methods, since it leads to the same form for the scattering matrix.

5. UNSTABLE PARTICLES

The asymptotic condition does not lend itself to any very simple treatment of unstable particles. Thus, it is of some interest to apply the above methods to this problem. We shall consider the simple case where a neutral scalar particle of mass M is unstable against decay to two stable particles, each of mass μ. The unstable particle is associated with a real scalar field $\phi^M(x)$. It is assumed that the mean life Γ^{-1} of the unstable particle is very long compared with the Compton wavelength M^{-1}. This condition is satisfied by most of the known elementary particles—the ratio of the two intervals is $M\Gamma^{-1}\sim 10^{15}$ for a charged pion—but not for particles such as the ρ meson which have been manifest only as resonances. Here, the mean life may be typically of the same order as the Compton wavelength.

As in the case of stable particles, we define the operator [Eq. (8)]

$$a_k{}^M(x^0)=i\int d^3x\frac{e^{-ik\cdot x}}{(2k_0)^{1/2}}\frac{\overleftrightarrow{\partial}}{\partial x^0}\phi^M(x), \quad (47)$$

where $k_0=(\mathbf{k}^2+M^2)^{1/2}$. The state vector

$$\int_{x^0-\Delta}^{x^0+\Delta}\frac{dx^{0\prime}}{2\Delta}a_k{}^{M\dagger}(x^{0\prime})|0\rangle, \quad (48)$$

where Δ satisfies the conditions

$$k_0/M^2\ll\Delta\ll k_0/\Gamma M, \quad (49)$$

[cf. Eq. (20)] is taken to describe a system which at time x^0 almost certainly will be found to consist of a single particle of mass M.

It is easy to verify, by the methods of Sec. 3, that with the condition (49) the mass of the state (48) almost certainly is equal to M, where the uncertainty in mass would be small compared with M, but large compared with Γ, if the weight function (6) for $\phi^M(x)$ were sufficiently smooth. However, the weight function

is expected to exhibit a resonance with width Γ at the mass M, so the uncertainty in mass actually is of the order of Γ.

The only aspect of the state vector (48) we shall consider is the question of the decay law for the system described by this vector. The probability amplitude that at time x^0+t the state (48) has not decayed is clearly

$$A=\int_{x^0-\Delta}^{x^0+\Delta}\frac{dy^0}{2\Delta}\int_{x^0-\Delta}^{x^0+\Delta}\frac{dy^{0\prime}}{2\Delta}\langle 0|a_k^M(y^0)a_k^{M\dagger}(y^0+t)|0\rangle. \quad (50)$$

Consider the matrix element

$$F(t)=\langle 0|a_k^M(y^0)a_{k'}^{M\dagger}(y^0+t)|0\rangle. \quad (51)$$

Using the definition (47), we have

$$F(t)=i^2\int d^3y d^3y'\frac{e^{-ik\cdot y}}{(2k_0)^{1/2}}\frac{\overleftrightarrow{\partial}}{\partial y^0}\langle 0|\phi_{(y)}{}^M\phi_{(y')}{}^M|0\rangle \times\frac{\overleftrightarrow{\partial}}{\partial y^{0\prime}}\frac{e^{ik'\cdot y'}}{(2k_0')^{1/2}}, \quad (52)$$

where $y^{0\prime}-y^0=t$, and $k_0=(\mathbf{k}^2+M^2)^{1/2}$, $k^{0\prime}=(\mathbf{k}'^2+M^2)^{1/2}$. But by the spectral representation (5),

$$\langle 0|\phi_{(y)}{}^M\phi_{(y')}{}^M|0\rangle=\sum_{\mathbf{k}^n}\int dk_0{}^n\,\rho^M(-k^{n2})e^{i(y-y')\cdot k^n}, \quad (53)$$

where

$$\rho^M(-k^{n2})=\sum_\alpha|\langle 0|\phi^M{}_{(0)}|n,\alpha\rangle|^2, \quad (54)$$

so that Eq. (52) may be written in the form

$$F(t)=\frac{\delta(\mathbf{k}-\mathbf{k}')}{2k_0}\int dk_0{}^n(k_0+k_0{}^n)^2 \times e^{i(k_0{}^n-k_0)t}\rho^M(k_0{}^{n2}-\mathbf{k}^2). \quad (55)$$

It is apparent from this equation that $F(t)$ vanishes in the limit of large t if the weight function $\rho^M(-k^2)$ has no delta function singularities. Assuming that this is the case, the value of $F(t)$ for sufficiently large t is determined by the behavior of the weight function for the argument near M^2 [recall $k_0=(\mathbf{k}^2+M^2)^{1/2}$]. To obtain the correct form for the decay law, we have to assume that the weight function is given by the approximate equation

$$\rho^M(k_0{}^{n2}-\mathbf{k}^2)=\frac{1}{\pi}\frac{\Gamma M}{(M^2+\mathbf{k}^2-k_0{}^{n2})^2+\Gamma^2M^2}, \quad (56)$$

where Γ is a constant, and where this equation is assumed to be valid if

$$|M^2+\mathbf{k}^2-k_0{}^{n2}|\ll M^2. \quad (57)$$

On rewriting the condition (57) in the form $|k_0-k_0{}^n|\ll M^2k_0{}^{-1}$, we see that this assumption is sufficient to determine $F(t)$ [Eq. (55)] if $t\gg k_0M^{-2}$. It will be convenient to use the approximate formula

$$(M^2+\mathbf{k}^2-k_0{}^{n2})^2+\Gamma^2M^2 =(k_0{}^2-k_0{}^{n2})^2+\Gamma^2M^2 \cong(2k_0)^2\left[(k_0-k_0{}^n)^2+\left(\frac{\Gamma M}{2k_0}\right)^2\right] \quad (58)$$

in Eq. (56). By the condition (57), this does not introduce any appreciable error.

Taking $t\gg k_0M^{-2}$, we use Eqs. (56) and (58) in (55), to get

$$F(t)\cong\frac{\delta(\mathbf{k}-\mathbf{k}')}{2k_0}\frac{\Gamma M}{\pi}\int_{-\infty}^{\infty}\frac{dk_0{}^n\,e^{i(k_0{}^n-k_0)t}}{(k_0-k_0{}^n)^2+(\Gamma M/2k_0)^2}. \quad (59)$$

On completing the contour in the upper-half $k_0{}^n$ plane, we obtain

$$F(t)\cong-\delta(\mathbf{k}-\mathbf{k}')e^{-\Gamma Mt/2k_0}, \quad (60)$$

where this is valid if

$$t\gg k_0M^{-2}.$$

This result should be substituted into the formula (50) for the probability amplitude. However, $\Delta\ll k_0\Gamma^{-1}M^{-1}$ [Eq. (49)] and the probability amplitude (50) is sensibly equal to $F(t)$ [Eq. (60)]. It is concluded that the probability that the system (48) has not decayed after a time interval t is

$$P(t)=\sum_{\mathbf{k}'}|\delta(\mathbf{k}-\mathbf{k}')e^{-\Gamma Mt/2k_0}|^2 =e^{-\Gamma Mt/k_0}. \quad (61)$$

This is the appropriate form for the decay law, where the rate of decay from rest is Γ. The factor $Mk_0{}^{-1}$ provides correct time dilation.

If t is not large compared with k_0M^{-2}, Eq. (60) is not valid. However, it is easy to verify that the probability amplitude (50) is equal to unity for $t\ll k_0\Gamma^{-1}M^{-1}$. This follows from Eqs. (55) and (56) and the condition (49). That is, (61) is valid for all values of t.

The formula (56) for the weight function might be indicated by the following argument. For the complete set of states in (54) we could use the set of asymptotic out states, $|k_1,k_2,\ldots k_{n,\text{out}}\rangle$, consisting of collections of free particles, each of mass μ, the ith particle having momentum k_i. Since the unstable particle of mass M decays to two particles of mass μ, we expect that

$$\langle 0|\phi_{(x')}{}^M\phi_{(x)}{}^M|0\rangle =\sum_{\mathbf{k}_1,\mathbf{k}_2}|\langle 0|\phi^M(0)|k_1,k_{2\text{out}}\rangle|^2e^{i(k_1+k_2)\cdot(x'-x)} \quad (62)$$

$$=\sum_{\mathbf{k}^n}\int dk_n{}^0\,e^{ik^n\cdot(x'-x)}\Big[\sum_{\mathbf{k}_1,\mathbf{k}_2}|\langle 0|\phi^M{}_{(0)}|k_1,k_{2\text{out}}\rangle|^2 \times\delta(k_1+k_2-k^n)\Big]. \quad (63)$$

Here $\delta(k_1+k_2-k^n)$ is supposed to be a Dirac delta function of energy and a Kronecker delta function of momentum. On comparing Eqs. (63) and (53), we find

$$\rho^M(-k^{n^2})=\sum_{\mathbf{k}_1,\mathbf{k}_2}|\langle 0|\phi^M{}_{(0)}|k_1,k_{2\text{out}}\rangle|^2\delta(k_1+k_2-k^n). \quad (64)$$

The matrix element in this equation has been considered previously, from the point of view of perturbation theory and of S-matrix methods.[4] For example, according to the ideas of perturbation theory, we would write

$$(8k_{10}k_{20})^{1/2}\langle k_1,k_{2\text{out}}|\phi^M{}_{(0)}|0\rangle = -g\Delta_F(-(k_1+k_2)^2)V(-(k_1+k_2)^2), \quad (65)$$

where g is the renormalized $M\mu\mu$ coupling constant, Δ_F is the renormalized propagator for the unstable particle, and V is the renormalized $M\mu\mu$ vertex function. For $-(k_1+k_2)^2\sim M^2$, we have $V\sim 1$ and

$$\Delta_F(-(k_1+k_2)^2)\cong 1/[(k_1+k_2)^2+M^2-i\Gamma M], \quad (66)$$

where Γ is the decay rate,

$$\Gamma=(g^2/16\pi)[(M^2-4\mu^2)^{1/2}/M^2]. \quad (67)$$

Using Eqs. (65) and (66), Eq. (64) becomes

$$\rho^M(-k^{n^2})\cong\frac{g^2}{(M^2+k^{n2})^2+\Gamma^2M^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\frac{\delta(k_1+k_2-k^n)}{8k_{10}k_{20}}, \quad (68)$$

where this equation is valid for $-k^{n^2}\sim M^2$. The right-hand side may be evaluated in the standard way,

$$\rho^M(-k^{n^2})\cong\frac{g^2}{16\pi^2}\frac{(M^2-4\mu^2)^{1/2}}{M}\frac{1}{(M^2+k^{n^2})^2+\Gamma^2\mu^2}$$
$$\cong\frac{1}{\pi}\frac{\Gamma M}{(M^2+k^{n^2})^2+\Gamma^2M^2}, \quad (69)$$

where the second line follows from Eq. (67). This is Eq. (56).

In this section the state vector for a system which at some given time consists of a single unstable particle has been constructed by the method used in the previous sections for the case of stable particles. It has been shown that the system decays with time if the weight function for the unstable particle field has no delta function singularities, and that if the weight function exhibits a resonance behavior at the energy of the unstable particle, the system decays according to the ordinary decay law. It was remarked that the required behavior of the weight function is indicated by the ideas of perturbation theory, as well as more general arguments.

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PHYSICAL REVIEW VOLUME 128, NUMBER 3 NOVEMBER 1, 1962

Behavior of Regge Poles in a Potential at Large Energy*

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Following Mandelstam's suggestion, we consider a potential which can be expanded in a power series in r, beginning with $1/r$. Then at large energy (positive or negative) there will be Regge poles at all negative integral values of l. If the $1/r$ term is absent in the expansion of the potential, the pole at $l=-1$ will be absent. Generally, if r^{2n-1} is the first nonvanishing *odd* power term in the potential expansion, the poles at $-1, -2, \cdots, -n$ are absent. If the potential expansion contains only even non-negative powers of r, there will be no Regge poles at finite negative l in the limit of large energy. If the potential behaves as $-a/r^2$ at small r, the scattering amplitude has a cut in the l plane from $-\frac{1}{2}-a^{1/2}$ to $-\frac{1}{2}+a^{1/2}$, and the Regge poles are no longer located at negative integers at large energy. For finite energy, it is shown that a pole at a positive integer or half-integer l implies another pole at $-l-1$ but that the residues at these poles are, in general, different from each other.

I

THE behavior of Regge poles in an ordinary, nonrelativistic potential has attracted considerable interest.[1] In particular, their behavior for large energy, positive or negative, may be important because of the possible analogy with the scattering problem for highly relativistic particles.[2] Mandelstam[3] has given a simplified theory of Regge poles in an ordinary potential which we shall use as a starting point of our calculations.

* Supported in part by the joint program of the Office of Naval Research and the U. S. Atomic Energy Commission.

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