Here $\delta(k_1+k_2-k^n)$ is supposed to be a Dirac delta function of energy and a Kronecker delta function of momentum. On comparing Eqs. (63) and (53), we find

$$\rho^M(-k^{n^2})=\sum_{\mathbf{k}_1,\mathbf{k}_2}|\langle 0|\phi^M{}_{(0)}|k_1,k_{2\text{out}}\rangle|^2\delta(k_1+k_2-k^n). \quad (64)$$

The matrix element in this equation has been considered previously, from the point of view of perturbation theory and of S-matrix methods.[4] For example, according to the ideas of perturbation theory, we would write

$$(8k_{10}k_{20})^{1/2}\langle k_1,k_{2\text{out}}|\phi^M{}_{(0)}|0\rangle \\ =-g\Delta_F(-(k_1+k_2)^2)V(-(k_1+k_2)^2), \quad (65)$$

where g is the renormalized $M\mu\mu$ coupling constant, Δ_F is the renormalized propagator for the unstable particle, and V is the renormalized $M\mu\mu$ vertex function. For $-(k_1+k_2)^2\sim M^2$, we have $V\sim 1$ and

$$\Delta_F(-(k_1+k_2)^2)\cong 1/[(k_1+k_2)^2+M^2-i\Gamma M], \quad (66)$$

where Γ is the decay rate,

$$\Gamma=(g^2/16\pi)[(M^2-4\mu^2)^{1/2}/M^2]. \quad (67)$$

Using Eqs. (65) and (66), Eq. (64) becomes

[4] M. Gell-Mann and F. Zachariasen, Phys. Rev. **124**, 953 (1961).

$$\rho^M(-k^{n^2})\cong\frac{g^2}{(M^2+k^{n2})^2+\Gamma^2M^2}\sum_{\mathbf{k}_1,\mathbf{k}_2}\frac{\delta(k_1+k_2-k^n)}{8k_{10}k_{20}}, \quad (68)$$

where this equation is valid for $-k^{n^2}\sim M^2$. The right-hand side may be evaluated in the standard way,

$$\rho^M(-k^{n^2})\cong\frac{g^2}{16\pi^2}\frac{(M^2-4\mu^2)^{1/2}}{M}\frac{1}{(M^2+k^{n^2})^2+\Gamma^2\mu^2} \\ \cong\frac{1}{\pi}\frac{\Gamma M}{(M^2+k^{n^2})^2+\Gamma^2M^2}, \quad (69)$$

where the second line follows from Eq. (67). This is Eq. (56).

In this section the state vector for a system which at some given time consists of a single unstable particle has been constructed by the method used in the previous sections for the case of stable particles. It has been shown that the system decays with time if the weight function for the unstable particle field has no delta function singularities, and that if the weight function exhibits a resonance behavior at the energy of the unstable particle, the system decays according to the ordinary decay law. It was remarked that the required behavior of the weight function is indicated by the ideas of perturbation theory, as well as more general arguments.

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Behavior of Regge Poles in a Potential at Large Energy*

H. A. Bethe and T. Kinoshita
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
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Following Mandelstam's suggestion, we consider a potential which can be expanded in a power series in r, beginning with $1/r$. Then at large energy (positive or negative) there will be Regge poles at all negative integral values of l. If the $1/r$ term is absent in the expansion of the potential, the pole at $l=-1$ will be absent. Generally, if r^{2n-1} is the first nonvanishing *odd* power term in the potential expansion, the poles at $-1, -2, \cdots, -n$ are absent. If the potential expansion contains only even non-negative powers of r, there will be no Regge poles at finite negative l in the limit of large energy. If the potential behaves as $-a/r^2$ at small r, the scattering amplitude has a cut in the l plane from $-\frac{1}{2}-a^{1/2}$ to $-\frac{1}{2}+a^{1/2}$, and the Regge poles are no longer located at negative integers at large energy. For finite energy, it is shown that a pole at a positive integer or half-integer l implies another pole at $-l-1$ but that the residues at these poles are, in general, different from each other.

I

The behavior of Regge poles in an ordinary, nonrelativistic potential has attracted considerable interest.[1] In particular, their behavior for large energy, positive or negative, may be important because of the possible analogy with the scattering problem for highly relativistic particles.[2] Mandelstam[3] has given a simplified theory of Regge poles in an ordinary potential which we shall use as a starting point of our calculations.

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[1] T. Regge, Nuovo cimento **14**, 951 (1959); **18**, 947 (1960); A. Bottino, A. M. Longoni, and T. Regge, *ibid.* **23**, 954 (1962).

[2] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962); S. C. Frautschi, M. Gell-Mann, and F. Zachariasen, *ibid.* **126**, 2204 (1962).

[3] S. Mandelstam, University of Birmingham, 1962 (to be published).

Our purpose is to clarify and, in some respects, modify some of Mandelstam's conclusions.

We consider the Schrödinger equation for large *negative* energy $E=k^2=-\gamma^2$:

$$\psi''(r)-(\gamma^2+l(l+1)/r^2)\psi(r)=V(r)\psi(r). \quad (1)$$

With Mandelstam, we assume that the power series expansion of the potential

$$V(r)=-A_0/r-A_1-A_2r-\cdots \quad (2)$$

converges for all r, and that rV is bounded for $r\rightarrow+\infty$. A superposition of Yukawa potentials, of exponentials, or of Gaussians may satisfy these criteria, but a potential like $(r+a)^{-2}$ does not because its power series does not converge for $r>a$. Discontinuous potentials like a square well are also excluded.

We want to emphasize here that the assumption (2) is indeed a very severe restriction on the potential $V(r)$. For instance, a superposition of Yukawa potentials,

$$V(r)=\int_{\mu_0^2}^{\infty}d\mu^2\,\sigma(\mu^2)\frac{e^{-\mu r}}{r}, \quad (3)$$

satisfies (2) only if

$$\int_{\mu_0^2}^{\infty}d\mu^2\sigma(\mu^2)\mu^n<\infty \text{ for all non-negative integer } n. \quad (4)$$

This is why Mandelstam is able to prove more far-reaching results than Regge[1] and Froissart[4] who take a more general approach.

Once (2) is assumed, we may solve (1) by the time-honored polynomial method. We first separate out the asymptotic behavior

$$\psi(r)=e^{-\gamma r}\phi(r), \quad (5)$$

and, setting $x=2\gamma r$, obtain

$$\frac{d^2\phi}{dx^2}-\frac{d\phi}{dx}-\frac{l(l+1)}{x^2}\phi=v\phi, \quad (6)$$

where

$$v=-a_0/x-a_1-a_2x-\cdots, \quad (7)$$
$$a_n=A_n(2\gamma)^{-n-1}.$$

Assuming that ϕ can be expanded as

$$\phi=\sum_{\nu=0}^{\infty}c_\nu x^{\nu+s}, \quad (8)$$

we obtain the recursion formula

$$[(\nu+1+s)(\nu+s)-l(l+1)]c_{\nu+1}-(\nu+s)c_\nu+\sum_{n=0}^{\nu}a_nc_{\nu-n}=0. \quad (9)$$

[4] M. Froissart, Princeton University, 1962 (to be published).

This equation can be satisfied for $\nu=-1$ only if s is equal to $l+1$ or $-l$. If we treat l as a variable, one case may be obtained from the other by analytic continuation. For convenience, let us choose $s=l+1$ in (9) where l may be positive or negative. Then we obtain the recursion formula for the c_ν:

$$(\nu+1)(\nu+2l+2)c_{\nu+1}=(\nu+l+1)c_\nu-\sum_{n=0}^{\nu}a_nc_{\nu-n}. \quad (10)$$

This equation can be solved successively for c_1, c_2, $\cdots$ insofar as $\nu+2l+2\neq0$. Thus, each c_ν is obviously meromorphic in the entire l-plane. Since the series (8) converges uniformly, the wave function ψ is meromorphic in l.[5]

If the power series (8) terminates, a Regge pole will certainly exist, because then ψ will behave as $e^{-\gamma r}$ for large r, which is the condition for a Regge pole. In general, however, we shall find that the series does not exactly terminate. Nevertheless we can, by proper choice of l, make the higher coefficients as small as we like, such that they decrease much faster than $1/\nu!$.[6] This is sufficient for a Regge pole. It is also necessary because, for a general l, the recursion formula (10) shows that c_ν behaves as $1/\nu!$ so that ϕ tends to $e^{2\gamma r}$ and no bound solution is obtained.

II

For large energy $-\gamma^2$, discussion of (10) is greatly facilitated by the fact that the coefficients a_n of the potential are small of order γ^{-n-1}, as is seen from (7). In this section, we assume that $a_0\neq0$, i.e., that the potential has a $1/r$ singularity. Setting $c_0=1$ and using the recursion formula (10), we find that all c_ν with $\nu\leq-l-1$ are of the order of magnitude 1.[6] But then, for ν approximately equal to $-l-1$, the right-hand side of (10) becomes very small for large γ. Calling m the non-negative integer nearest to $-l-1$, we find the right-hand side for $\nu=m$ proportional to

$$l+m+1-a_0-\sum_{n=1}^{m}(c_{m-n}/c_m)a_n. \quad (11)$$

Since the ratios c_{m-n}/c_m are all of order unity, the sum over n is of order γ^{-2} according to (7). Thus, if we choose l such that

$$l=-m-1+a_0+O(\gamma^{-2}), \quad (12)$$

we find that (11), and hence c_{m+1}, is of order γ^{-2} or less.

Let us next consider the recursion formula for c_{m+2}:

$$(m+2)(m+2l+3)c_{m+2}=(m+l+2-a_0)c_{m+1}-a_1c_m-\cdots. \quad (13)$$

Since $a_1\sim\gamma^{-2}$ and $c_m\sim1$, we find that c_{m+2} is of order

[5] The asymptotic behavior of ϕ in l may be examined using the generalized WKB method of reference 1.

[6] Under normal circumstances, c_ν behaves as $1/\nu!$. In this paper, c_ν will thus be considered as large or small according as $\nu!c_\nu\gtrsim1$ or $\ll1$.

γ^{-2}. All subsequent c_ν, up to $\nu=2m$, will then also be of order γ^{-2}. Applying (10) to $\nu=2m$, we get

$$c_{2m+1}=\frac{l+2m+1-a_0}{(2m+1)2(l+m+1)}c_{2m}+\text{terms of relative order }\gamma^{-2}. \quad (14)$$

This has the small denominator $l+m+1\approx -a_0=O(\gamma^{-1})$. Since we have shown that $c_{2m}=O(\gamma^{-2})$, we find that $c_{2m+1}=O(\gamma^{-1})$. All subsequent terms of the series will also be of this order. Denoting $\sum_{\nu=0}{}^{m} c_\nu x^{\nu+l+1}$, $\sum_{m+1}{}^{2m} c_\nu x^{\nu+l+1}$, and $\sum_{2m+1}{}^{\infty} c_\nu x^{\nu+l+1}$ by ϕ_1, ϕ_2, and ϕ_3, respectively, we may summarize the above result as follows:

$$\phi_1=O(1),\quad \phi_2=O(\Delta l-a_0)=O(\gamma^{-2}),$$
$$\phi_3=O((\Delta l-a_0)/\Delta l)=O(\gamma^{-1}), \quad (15)$$

where $\Delta l=l+m+1=O(\gamma^{-1})$. Thus, provided $a_0\neq 0$, all c_ν for $\nu>m$ are made arbitrarily small in the limit of large γ. Since the series (8) is thus effectively terminated, we may conclude that there is a Regge pole in the γ^{-2} neighborhood of $l=-m-1+a_0$.

We can improve this argument as follows: By making the expression (11) equal to a suitable multiple of a_1, we can cancel the first two terms on the right-hand side of (13), and make c_{m+2} of order γ^{-3} rather than γ^{-2}, which will then make c_{2m+1} of order γ^{-2}. This leaves the statement (12) unchanged. By successive consideration of the recursion formula for $\nu=m+2$, $m+3$, etc., and appropriate adjustment of l, we can make the coefficients c_ν for large ν small of order γ^{-n} with arbitrarily large n. In this manner we can show that there is a value of l for which the function ψ will not contain an exponentially increasing component, i.e., there is a Regge pole near $l=-m-1+a_0$.

For $\gamma\rightarrow\infty$, the Regge poles will be at the negative integers. For finite but large γ, their position is given by (12).[7] The poles are thus determined by the $1/r$ part of the potential, as stated by Mandelstam, and are equally spaced to order γ^{-2}.

III

The situation is somewhat different if $a_0=0$, i.e., if the potential is regular at the origin. Then choosing l to satisfy (12) will make $l+m+1=O(\gamma^{-2})$. As a consequence, (14) yields c_{2m+1} of order 1, since $c_{2m}=O(\gamma^{-2})$. This is also seen from the behavior of ϕ_3 in (15). Thus, the argument of Sec. II does not make the high ν terms of the series small and a more accurate discussion is needed. That this can be a serious change is shown by considering the case $m=0$, i.e., the Regge pole near $l=-1$. Let us first note that the recursion formula (10) gives, for $a_0=0$ and $\nu=0$,

$$c_1=[(l+1)/(2l+2)]c_0=\tfrac{1}{2}c_0=O(1), \quad (16)$$

independent of the potential and of l. Since l is assumed to be near -1, there is no value of $\nu>0$ for which the right-hand side of (10) could vanish approximately for large γ. Therefore, the series will continue with large coefficients to arbitrarily large ν, and ψ will behave as $e^{+\gamma r}$, hence, we do not get a Regge pole at -1.

If $m\neq 0$, we still can get a Regge pole. We must apply the recursion relation (10) for several values of ν near m, to make the high c_ν sufficiently small. It turns out that we must consider (10) for $\nu=m-2$ up to $\nu=m+2$. If we neglect $l+m+1$ whenever it is added to an integer of order unity or larger, and express all c_ν in terms of c_m, we obtain

$$c_{m-1}=m(m+1)c_m-a_1c_{m-2}-\cdots, \quad (17a)$$

$$c_{m-2}=\tfrac{1}{2}(m-1)m(m+1)(m+2)c_m-\tfrac{1}{2}a_1c_{m-3}-\cdots, \quad (17b)$$

$$m(m+1)c_{m+1}=-(l+m+1)c_m+a_1c_{m-1}+a_2c_{m-2}+\cdots, \quad (17c)$$

$$(m-1)(m+2)c_{m+2}=-c_{m+1}+a_1c_m+a_2c_{m-1}+\cdots, \quad (17d)$$

$$(m-2)(m+3)c_{m+3}=-2c_{m+2}+a_1c_{m+1}+a_2c_m+\cdots. \quad (17e)$$

According to (17c), c_{m+1} and all subsequent c_ν are small at least of order γ^{-2} (if $c_m=1$), hence the term $a_1c_{m+1}(\sim\gamma^{-4})$ in (17e) is negligible compared to the other two terms. Inserting (17b) and (17a) into (17c) gives, with $c_m=1$,

$$c_{m+1}=-(l+m+1)/m(m+1)+a_1+\tfrac{1}{2}(m-1)(m+2)a_2+\cdots. \quad (18a)$$

Inserting this into (17d), we note that a_1 cancels. We have already seen that a_1c_{m+1} in (17e) (and subsequent equations) may be neglected; hence, we find that a_1 has no influence on the high c_ν's. Therefore, a_1 by itself cannot give us a Regge pole solution. This will be confirmed in a more general context in the next section.

Inserting (18a) into (17d), (17e) gives

$$(m-2)(m-1)(m+2)(m+3)c_{m+3}=-2(l+m+1)/m(m+1)-4a_2+O(a_3). \quad (18b)$$

This can be made small of order γ^{-4} if we set

$$l=-m-1-2m(m+1)a_2 \quad (19)$$

so that l differs from an integer in order γ^{-3}. By this choice, c_{m+2} will be of order γ^{-3}, c_{m+1} of order γ^{-2}, and in the recursion formula

$$(m-3)(m+4)c_{m+4}=-3c_{m+3}+a_1c_{m+2}+a_2c_{m+1}+a_3c_m+\cdots, \quad (20)$$

the first term is of order γ^{-4}, the next two terms are of order γ^{-5}, and a_3c_m is again of order γ^{-4}. Hence, all further terms up to $\nu=2m$ are of order γ^{-4}. At $\nu=2m+1$ we get a denominator $2(l+m+1)$ which, according to (19), is proportional to $a_2=O(\gamma^{-3})$. Hence, c_{2m+1} is of order γ^{-1} and so are all subsequent terms. Thus, in the

[7] Actually, the correction term in (14) is of order γ^{-3} as can be shown by an argument similar to that of Sec. III.

limit $\gamma \to \infty$ the series has been successively terminated, and the termination can be improved by adding to (19) a suitable term of order γ^{-4}, etc. There is, therefore, a Regge pole at l given by an expression close to (19), *except* for $m=0$.

The cases $m=1$ and 2 require special investigation. Thus, if $m=2$, (17e) and (18b) should have $-l-3$ on the left-hand side rather than $m-2$. But our argument was based on making the right-hand side of (18b) of order γ^{-4} which will then make $c_{m+3}(=c_{2m+1})$ and all subsequent c_ν of order γ^{-1} as before. For $m=1$, we have, of course, $c_{m-2}=0$, and in (17d) the factor $m-1$ is replaced by the small but finite $-l-2$. Then, as can be shown by explicit calculation, if we choose l according to (19) plus a term of order γ^{-5}, c_3 and all subsequent terms will be of order γ^{-2}, which is even smaller than their value in the general case.

The above argument must be modified if $a_2=0$ ($a_1\neq 0$) in addition to $a_0=0$. In this case, we may try to make c_{m+3} small of order γ^{-4} by choosing $l+m+1=O(\gamma^{-4})$. However, since $l+m+1$ occurs in the denominator of the recursion formula for $\nu=2m$, the terms $\nu\geq 2m+1$ will now be of order $(l+m+1)/(l+m+1)=O(1)$. Thus, the series will not terminate for $\gamma\to\infty$ and the above consideration will not succeed. Obviously this situation is very similar to that discussed at the beginning of this section, and can in fact be treated in the same manner. Thus, we find that we do not get Regge poles at -1 and -2, but we still find Regge poles at $-3, -4, \cdots$.[8]

In the same manner, we expect that if a_{2n} is the first nonvanishing even term of the potential (7), the Regge poles at $-1, -2, \cdots, -n$ are absent but there are poles near $-n-1$ and all negative integers beyond, and that the distance from the integer is of order $a_{2n}=O(\gamma^{-2n-1})$.

IV

We have seen in Sec. III that a constant term in the potential does not cause a Regge pole. We shall now show that there is in general no Regge pole at any finite, real, negative l if the potential consists of only even powers of r.

For this purpose it is better to use the original differential equation (1) directly, rather than make the substitution (5). We use here $y=\gamma r$ and write

$$v=-b_0-b_1y^2-b_2y^4-\cdots \tag{21}$$

$$b_n=A_{2n+1}\gamma^{-2n-2}. \tag{22}$$

Then if we expand ψ in a power series

$$\psi=\sum_{\nu=0}^{\infty} f_\nu y^{\nu+s}, \tag{23}$$

we get, for $s=l+1$,

$$(\nu+2)(2l+\nu+3)f_{\nu+2}=f_\nu-\sum_{n=0}^{\nu/2} b_n f_{\nu-2n}, \tag{24}$$

so that only even ν will occur.[9] Now the right-hand side does not have any coefficients depending on ν, in contrast to (10). Since the b_n are very small for very large γ according to (22), the right-hand side is dominated by f_ν. Hence the f_ν are all of the same sign for $\nu>|2l|$, and the asymptotic behavior of f_ν for large ν is that of the exponential series e^y which does not lead to a Regge pole in our case of large γ.

For any given *finite* γ, however, it is possible to have Regge poles at sufficiently large l. To see this, consider ν which is of the same order of magnitude as $-l$. Then, disregarding the sum in (24), we get

$$f_\nu\approx -f_{\nu-2}/l^2. \tag{25}$$

Thus, for large l, $|f_{\nu-2n}|\gg|f_\nu|$ so that $b_nf_{\nu-2n}$ can be of the same order as f_ν, even though b_n is small. Take, for instance, the Gaussian potential

$$V=-Ae^{-\mu^2r^2}. \tag{26}$$

Then (21) gives

$$b_n=(A/n!\gamma^2)(-\mu^2/\gamma^2)^n. \tag{27}$$

We therefore obtain

$$\sum_{n=0}^{\nu/2} b_nf_{\nu-2n}\approx A\gamma^{-2}f_\nu\exp(\mu^2l^2/\gamma^2), \tag{28}$$

making use of (25). Although $A/\gamma^2\ll 1$, the exponential will make this term larger than f_ν if l is sufficiently large, and we, therefore, may be able to make the sign of f_ν alternate. Thus, there will probably be Regge poles for[10]

$$|l|>(\gamma/\mu)[\ln(\gamma^2/A)]^{1/2}. \tag{29}$$

As $\gamma\to\infty$, these poles go to (negative) infinite l. It is true that, when (29) holds, the sum $\sum b_nf_{\nu-2n}$ of (24) cannot be disregarded in obtaining (25). However, we have convinced ourselves that this will not affect the qualitative features of our argument.

Returning to the case of very large γ and small l, we may inquire why the series (23) necessarily tends to e^{+y} while in the presence of odd powers of y in $v(y)$ the wave function may tend to e^{-y}. If we add to (21) a term a_0/y, we may still expand ψ in a power series (23). But the term a_0/y in the potential will now generate terms of odd ν in the series, and the recursion formula for $\nu=2m-1$ (m=integer nearest to $-l-1$) reads

$$(2m+1)2(l+m+1)f_{2m+1}=f_{2m-1}-a_0f_{2m}-\sum_{n=0}^{m-1} b_nf_{2m-1-2n}. \tag{30}$$

[8] We are greatly indebted to S. C. Frautschi for pointing out to us that the pole at -2 is absent when $a_0=0$, $a_1\neq 0$, and $a_2=0$.

[9] For $l=-1$, f_1 may be nonzero and thus may generate a series in odd ν. However we know already that there is no Regge pole at $l=-1$.

[10] Existence of Regge poles for the potential (26) is obvious since it certainly accommodates bound states if A is large.

Now f_{2m-1} (and all f of odd order $<2m-1$) will be of order a_0. If we choose $l+m+1=O(a_0)=O(\gamma^{-1})$, we obtain

$$f_{2m+1}/f_{2m}=O(a_0/l+m+1)=O(1). \qquad (31)$$

For any $\nu>2m$, the ratio

$$f_{2n+1}/(f_{2n}f_{2n+2})^{1/2} \quad (n=\text{integer larger than } m) \qquad (32)$$

will be of the same order as (31). By making this ratio (32) asymptotically tend to -1, we can obtain a series similar to that for e^{-y} and thus a Regge pole. This shows that it is not the method used in this section, viz. direct solution of the equation for ψ, which is responsible for the absence of Regge poles for the potential (21).

The most important potential expandable in even powers of r, and vanishing at ∞, is the Gaussian (26). Of sourse, this may be multiplied by any polynomial in r^2. Functions like $e^{-\alpha r^4}$ are further examples.

V

Mandelstam has also discussed the situation at *finite* energy k^2 where there is a Regge pole at l close to the integer (or half-integer) $m\geq 0$. There is then also a pole near $-m-1$. We want to discuss this problem, including the residues, in more detail. For simplicity and coherence with previous sections, we still assume the energy to be negative, $-\gamma^2$, but we now assume it to be finite.

Suppose there is, for fixed energy $-\gamma^2$, a Regge pole at $l=m+\alpha$ (m a non-negative integer or half-integer, $|\alpha|\ll 1$). This means that the power series (8) converges for large x to a result much smaller than e^x, so that (with proper normalization)

$$\psi(m+\alpha,x)\rightarrow e^{-x/2} \qquad (33)$$

(more accurately, $\psi\rightarrow x^{a_0}e^{-x/2}$). Then, for some $l=m+\beta$ close by, the wave function will be asymptotically

$$\psi(m+\beta,x)\rightarrow e^{-x/2}-B(\beta-\alpha)e^{x/2}, \qquad (34)$$

where B is a constant independent of β. Let us now consider the recursion formula (10) for $l=-m-1+\beta$ and $\nu=2m$ (which is an integer), and assume for the moment that its right-hand side does not vanish for $l=-m-1$ (i.e., $\beta=0$). We find easily that

$$c_{2m}\approx D_1\beta c_{2m+1}, \qquad (35)$$

where D_1 is again a constant. Likewise, all c_ν for $\nu<2m$ will be of order βc_{2m+1}. Thus, in the recursion formula (10) for $\nu=2m+1$

$$(2m+2)(1+2\beta)c_{2m+2} = (m+1)c_{2m+1}-a_0c_{2m+1}-a_1c_{2m}-\cdots, \qquad (36)$$

the terms c_{2m}, c_{2m-1}, etc., are all of relative order β. (The a_n are now of order unity, no longer small.) Furthermore, for small β, the recursion formula (10) for $l=-m-1+\beta$ and for any $\nu=2m+1+\nu'$, $\nu'>0$, is the same as that for $l=m+\beta$ and $\nu=\nu'$ to the lowest order in β. We thus obtain

$$\frac{c_{2m+1+\nu'}}{c_{2m+1}}(l=-m-1+\beta) =\frac{c_{\nu'}}{c_0}(l=m+\beta)+D(\nu')\beta, \quad \nu'>0, \qquad (37)$$

where $D(\nu')$ is a coefficient depending on ν' (and γ) but not on β to the lowest order. Then, since $D(\nu')$ in general behaves roughly as $1/\nu'!$, the asymptotic behavior of the wave function will be

$$\psi(-m-1+\beta,x)\rightarrow e^{-x/2}-[B(\beta-\alpha)+C\beta]e^{x/2}, \qquad (38)$$

where C is another constant. We therefore get a second Regge pole at $l=-m-1+\beta_0$, where

$$\beta_0=[B/(B+C)]\alpha. \qquad (39)$$

Each positive Regge pole near m (integer or half-integer) is accompanied by a negative one near $-m-1$ (but not vice versa).

If we normalize the wave function $\psi(m+\alpha,x)$ by (33), the coefficients c_ν in its expansion (8) will be of order unity.[6] The expansion coefficients c_ν for $l=-m-1+\beta$ will be the same as for $l=m+\alpha$, except for terms proportional to $\beta-\alpha$ or to β, both of which quantities are of order α, as is seen from (39). The behavior of the S matrix near the two Regge poles is

$$S=\frac{\text{coefficient of } e^{-x/2}}{\text{coefficient of } e^{x/2}}=\frac{1}{B(\beta-\alpha)} \quad \text{for} \quad l=m+\beta, \qquad (40)$$

$$=\frac{1}{(B+C)(\beta-\beta_0)} \quad \text{for} \quad l=-m-1+\beta. \qquad (41)$$

If we now let γ^2 vary, the positive pole will go exactly to $l=m$ at some value of the energy, $-\gamma_0^2$. Then the negative pole will go to $l=-m-1$ at exactly the same energy, according to (39). This is in accord with the result of Mandelstam. However, the residues will, in general, be different from each other, being $1/B$ and $1/(B+C)$, respectively.

In the case of a pure Coulomb potential, a_1, a_2, $\cdots$ all vanish in (36). This means that, instead of (37), we get for $\nu'\gg m$:

$$\frac{c_{2m+1+\nu'}}{c_{2m+\nu'}}(l=-m-1+\beta) =\frac{c_{\nu'}}{c_{\nu'-1}}(l=m+\beta)[1+O(\beta/\nu'^2)]. \qquad (37a)$$

Therefore the coefficients for large ν' tend to the same limit for positive and negative l, and thus $C=0$ in (38). Hence, the residues at $l=m$ and at $l=-m-1$ are identical in this case.[11] However, the next terms in the

[11] V. Singh, Phys. Rev. **127**, 632 (1962).

expansion of the S matrix at these poles need not be the same.

Assuming there is a positive Regge pole at $l=m$ for energy $-\gamma_0^2$, and calculating the right-hand side of the recursion formula (10) for $l=-m-1$ and $\nu=2m$, we shall find this, in general, to be a finite multiple of c_{2m}. However, if we now change the potential continuously, adjusting the energy to stay with the same Regge pole at $l=m$, we may find for some potentials V^* that this right-hand side will vanish. In this case we obtain $c_{2m}=K_1c_{2m+1}$, where K_1 is a finite number determined by the potential, instead of (35). All c_ν for $\nu\leq 2m+1$ are therefore of the same order of magnitude. This means that $C\beta$ of (38) is now of order 1 rather than β. Thus, for exceptional potentials V^*, one negative Regge pole may be absent. We may regard this as the limit where the residue at $l=-m-1$ vanishes. In fact, if we consider a potential $V^*+\Delta V$, where ΔV is small, we shall find that the coefficient C of (38) becomes larger and larger as ΔV decreases. As a consequence, the residue $1/(B+C)$ will become smaller and smaller and will vanish in the limit $\Delta V=0$.

As was pointed out, this behavior of Regge poles at $l=m$ and $l=-m-1$ occurs for both integer and half-integer l because only the recursion formula for $\nu=2m$ matters. [See (35).] On the other hand, in Secs. II and III we showed that the Regge poles for large energy go to negative integers, not half-integers: in that case, the recursion formula for $\nu=m$ was the important one.

VI

The simple method of Mandelstam is unfortunately not applicable to more general potentials. An exception is the potential

$$v=-a/x^2-a_0/x-a_1-a_2x-\cdots, \quad (42)$$

which behaves as $1/x^2$ at the origin. As is seen from (3), this may be regarded as a particular superposition of Yukawa potentials with a mass spectrum which behaves as $\sigma(\mu^2)\to 1/\mu$ for large μ. Since the term a/x^2 has the same x dependence as the centrifugal barrier term, this problem can be reduced to the case (7) by the transformation

$$[(l+\tfrac{1}{2})^2-a]^{1/2}-\tfrac{1}{2}\to l. \quad (43)$$

Thus, all results of previous sections may be translated without difficulty to this case and no new problem arises. Nevertheless, it will be useful to examine the analyticity in l of this scattering amplitude since it may give us some insight in the behavior of Regge poles when the potential is more singular than simple Yukawa potentials.

As is easily seen, the recursion formula (9) now becomes

$$[(\nu+1+s)(\nu+s)-l(l+1)+a]c_{\nu+1}-(\nu+s)c_\nu+\sum_{n=0}^{\nu}a_nc_{\nu-n}=0. \quad (44)$$

For $\nu=-1$, this equation can be satisfied if and only if

$$s=\tfrac{1}{2}\pm[(l+\tfrac{1}{2})^2-a]^{1/2}. \quad (45)$$

Let us choose $s=1/2+[(l+1/2)^2-a]^{1/2}$ following the convention of Sec. II. Then (45) becomes

$$(\nu+1)(\nu+1+2[(l+\tfrac{1}{2})^2-a]^{1/2})c_{\nu+1}=(\nu+\tfrac{1}{2}+[(l+\tfrac{1}{2})^2-a]^{1/2})c_\nu-\sum_{n=0}^{\nu}a_nc_{\nu-n}, \quad (46)$$

which reduces to the form (10) by the substitution (43) as was noted above. Obviously, c_ν is meromorphic in the l plane cut along the line connecting the two fixed branch points

$$l=-\tfrac{1}{2}-a^{1/2},\quad l=-\tfrac{1}{2}+a^{1/2}. \quad (47)$$

The cut is so chosen that $[(l+1/2)^2-a]^{1/2}$ changes its sign from positive to negative as l moves in the cut l plane from a large positive value to a large negative value. The cut lies on the real axis if $a>0$ (attractive at small distance)[12] and on the vertical line $\mathrm{Re}l=-1/2$ if $a<0$ (repulsive at small distance).

As is seen from (43), if a Regge pole for the case $a=0$ is located at $l=l_0$, the corresponding Regge pole for $a\neq 0$ will be found at

$$l=-\tfrac{1}{2}+[(l_0+\tfrac{1}{2})^2+a]^{1/2}. \quad (48)$$

Thus, if l_0 is real and $>-1/2$, the Regge pole will shift to the right of l_0 as a increases from zero (attractive short-range force), while it will shift to the left as a decreases from zero (repulsive short-range force). This is in accord with the expectation that, for a given energy, more attractive potentials will accommodate bound states of higher angular momenta than those of less attraction. On the other hand, if l_0 is real and $<-1/2$, Regge poles move to the left or right according as $a>0$ or $a<0$, somewhat contrary to the common sense argument. However, it is easy to understand this behavior: It is because the horizontal cut pushes out the complex l plane along both the positive and the negative axis while the vertical cut deforms the l plane in the opposite sense.

In the limit $\gamma\to\infty$ where l_0 approaches one of the integers $-1, -2, \cdots$, the Regge pole (48) is found on the real axis as long as $a>-1/4$. When a decreases beyond $-1/4$, the Regge pole corresponding to $l_0=-1$ goes off the real axis and proceeds along the cut at

[12] For $a>1/4$, the cut extends into the region $\mathrm{Re}l>0$. This must be related to the well-known fact that $a=1/4$ represents the strongest attractive $1/r^2$ potential that can be handled by the usual scattering theory. Incidentally, in the Klein-Gordon equation with a Coulomb potential, the quadratic potential term $\pm e^2/r^2$ is always attractive and thus belongs to the case $a>0$. See V. Singh, reference 11.

Re$l=-1/2$. We find that this Regge pole follows the movement of the branch point (47) as a decreases, but cannot overtake it as long as $\gamma=\infty$. As a decreases further, more and more Regge poles line up on the cut line along Re$l=-1/2$.

As $E=-\gamma^2$ increases from $-\infty$, those Regge poles sitting on the real axis move to the right or left according as $a_0>0$ (attractive $1/r$ potential) or $a_0<0$ (repulsive $1/r$ potential). Regge poles along the vertical line Re$l=-1/2$ move upwards or downwards in an appropriate manner. When E approaches $+\infty$, Regge poles come back to the position given by (48) going through complex values. Of course, a more powerful method is required for the analysis of Regge trajectories for finite E, such as the continued fraction method of Lovelace.[13]

VII

For the potentials of the form (2), Regge poles tend to negative integers as γ approaches infinity. Apparently, the Regge pole at $l=-m-1$ ($m=0, 1, 2, \cdots$) is somehow related to the term r^{2m-1} of the potential $V(r)$. This may be better understood by comparison with the Born approximation which is known to be approached by the exact amplitude in the high-energy limit.[14] Regarding $V(r)$ as a superposition of "potentials" r^n rather than of Yukawa potentials, the Born scattering amplitude contributed by r^n is easily found to be

$$\int d^3x\, e^{i\Delta\mathbf{k}\cdot\mathbf{x}} r^{2m-1} \propto t^{-m-1}, \tag{49a}$$

$$\int d^3x\, e^{i\Delta\mathbf{k}\cdot\mathbf{x}} r^{2m} = 0, \tag{49b}$$

where $-t$ is the square of the momentum transfer. For $\gamma^2=-k^2\gg\mu^2$, an appropriate superposition of (49) for all m will converge to the scattering amplitude caused by $V(r)$ in the first Born approximation. The Regge pole expansion of the exact scattering amplitude,

$$\sum_i f_i(E) P_{l_i(E)}(-\cos\theta) \propto \sum_i f_i(E) t^{l_i(E)}[1+O(t^{-1})+\cdots], \tag{50}$$

gives a behavior as t^{l_1} for large t where Rel_1 is algebraically the largest Rel_i at a given E. Since in this limit the exact amplitude agrees with the Born approximation, comparison with (49) shows then that $l_1=-m-1$ if r^{2m-1} is the lowest odd power of r in the expansion of the potential. This is in exact agreement with our results in Sec. III.

[13] C. Lovelace, Imperial College, London (1962) (to be published).
[14] N. N. Khuri, Phys. Rev. **107**, 1148 (1957).

The relation (49b) shows further that the even power term r^{2m} ($m\geq 0$) plays no role in the determination of Regge poles in the large energy limit (although it may influence the location of poles at finite energies). This is related to the fact that, as a function of the coordinates x, y, z, r^{2m} is regular everywhere in the three-dimensional space which makes the Fourier transform (49b) vanish in the limit of high $\Delta\mathbf{k}$. (To see this still better, and avoid any convergence problem, take a function like $e^{-\alpha r^2}$, consider it as a product $e^{-\alpha x^2}e^{-\alpha y^2}e^{-\alpha z^2}$, and take the Fourier transform for each coordinate separately.) On the other hand, the origin $x=y=z=0$ is certainly a singularity for all terms r^{2m-1}. Note however that this singularity becomes smoother and smoother near the origin as m increases. This explains why the corresponding Regge poles move further and further away from the origin of the complex l plane.

When the energy is not large enough compared with the potential, the exact amplitude will not approach the first Born approximation, and thus the above argument will not work. If the potential contains a $1/r^2$ term, no energy is large enough compared with $V(r)$ since $1/r^2$ is of the same order of magnitude as the kinetic energy itself. Thus, the scattering amplitude does not converge to the Born approximation even at very large energy, and the corresponding Regge poles do not tend to negative integers as was mentioned in Sec. VI.

Mandelstam has shown how to improve the Watson-Sommerfeld representation when the scattering amplitude is meromorphic in the entire l plane and thus the path of the "background integration" along Re$l=-1/2$ can be pushed to Re$l=L$ where L is a finite but arbitrary real number smaller than $-1/2$. When there is a cut in addition to poles, his formula must be modified so that it includes the contribution from the part of the cut that lies on the right-hand side of the line Re$l=L$. If all Regge poles are found to the left of the cut, it is obviously difficult to approximate the scattering amplitude in terms of a few leading Regge poles. Such a situation will be found in the case of the singular potential (42). In general, if the potential is very singular at the origin, it will be necessary to disentangle the contribution of cuts before the effect of Regge poles is identified.

This paper is intended for the clarification of some mathematical problems concerning the Regge poles that arise from the special assumption (2) of Mandelstam about the potential $V(r)$. Although some of our results will be valid for a more general superposition of Yukawa potentials, others will undoubtedly be special consequences of special assumptions. In particular, it will be dangerous to speculate, on the basis of the results of nonrelativistic theory, about the behavior of Regge poles in relativistic particle physics.