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Possible Tests of the *TCP* Theorem in Proton-Antiproton Scattering*

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Some experiments in elastic proton-antiproton scattering are suggested in order to test the *TCP* theorem, as well as the validities of the separate *T*, *C*, and *P* invariances. Also, a theorem has been proved that the total scattering cross section of two particles is equal to that of antiparticles, provided that the *TCP* theorem is valid.

I. INTRODUCTION

It is generally believed that the strong interactions are invariant separately under the parity operation P, time reversal T, and charge conjugation C. Several experiments[1] show, indeed, that the parity is quite accurately conserved in nuclear reactions. In terms of the ratio F which gives a measure of parity-violating terms against parity-conserving terms for the matrix element of these reactions, the experiments set an upper limit of the order of 10^{-9} for F^2, though F is not unambiguously defined. As for the time-reversal invariance, several experiments[2] on various nuclear reactions place a limit of 10^{-3} on the square of the ratio of the time-reversal violating matrix element to the time-reversal conserving one. Finally, for the validity of the charge conjugation invariance, we may test it in the study of the reactions:

$$\bar{p}+p\rightarrow p+\bar{n}+\pi^-, \tag{1a}$$

$$\bar{p}+p\rightarrow n+\bar{p}+\pi^+, \tag{1b}$$

$$\bar{p}+p\rightarrow p+\bar{p}+\pi^0. \tag{1c}$$

The equality of the cross sections for the reaction Eq. (1a) and for the reaction Eq. (1b) suggests the validity of C or CP invariance as has been pointed out by Pais.[3] However, the present experimental data[4] are still too meager to give a sufficient accuracy, though they do not contradict the prediction of charge conjugation invariance. We may also note that the possible absence[5] of $\pi^0\rightarrow 3\gamma$ will give another piece of evidence for the validity of charge conjugation, but the present experiment is still not accurate enough to be able to say anything about it.

The purpose of this note is to point out that we can make a systematic study of the validities of P, C, T, and TCP invariances (or any combinations of these) in elastic proton-antiproton scattering:

$$\bar{p}+p\rightarrow\bar{p}+p. \tag{2}$$

Especially, we may be able to give a direct experimental test for the validity of the TCP theorem. Before going into details, let us briefly survey consequences of the validity of TCP invariance. If TCP invariance holds, then we obtain the following results:

(i) The masses of a particle and its antiparticle are the same[6] for stable particles. For unstable particles, we cannot define the mass, exactly speaking. However, if the instability is caused by weak interactions, then we can, in practice, define the mass by neglecting the weak interactions, without causing any theoretical and experimental difficulties. In that case, the above theorem holds also for unstable particles.

(ii) The magnetic moment of a particle is equal in magnitude, but with opposite sign, to that of its antiparticle.[7] The same is true for the electric dipole moment.[7]

(iii) For unstable particles the lifetimes of a particle and its antiparticle are the same,[6,8,9] at least in the lowest order of the weak interaction.

(iv) The total scattering cross section of two particles is the same as that of their antiparticles, if we can neglect the weak interactions.

The statement (iv) will be proved in the next section. Note that (iii) guarantees the equivalence of the total lifetime but not necessarily the equivalence of the partial lifetimes of a particle and its antiparticle.[9] Similarly, (iv) does not guarantee the equivalence of the partial cross sections for the scattering of two particles and of their antiparticles, unless C or CP invariance holds. Therefore, a comparison of partial cross sections may provide a test for the charge conjugation. One such example is the comparison of the

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[2] L. Rosen and J. E. Brolley, Phys. Rev. Letters **2**, 98 (1959); D. Bodansky, S. F. Eccles, G. W. Farwell, M. E. Rickey, and P. C. Robison, *ibid.* **2**, 101 (1959); A. Abashian and E. M. Hafner, *ibid.* **1**, 255 (1958); P. Hillman, A. Johannson, and G. Tibell, Phys. Rev. **110**, 1211 (1958); C. F. Hwang, T. R. Ophel, E. H. Thorndike, and R. Wilson, *ibid.* **119**, 352 (1960).

[3] A. Pais, Phys. Rev. Letters **3**, 242 (1959).

[4] N. H. Xuong, G. R. Lynch, and C. K. Hinricho, Phys. Rev. **124**, 575 (1961).

[5] R. P. Ely and D. H. Frisch, Phys. Rev. Letters **3**, 365 (1959).

[6] G. Luders and B. Zumino, Phys. Rev. **110**, 1450 (1958); and *Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics* (Interscience Publishers, Inc., New York, 1957), p. 55. R. E. Marshak and E. C. G. Sudarshan, *Introduction to Elementary Particle Physics* (Interscience Publishers, Inc., New York, 1961), p. 109.

[7] This can be proved in the same way as the proof of the nonexistence of an electric dipole moment of a particle when time-reversal invariance holds. See Marshak and Sudarshan, reference 6, p. 99.

[8] T. D. Lee, R. Oehme, and C. N. Yang, Phys. Rev. **106**, 340 (1957).

[9] S. Okubo, Phys. Rev. **109**, 984 (1958).

two reactions, Eq. (1a) and Eq. (1b), as has been mentioned already.

Now, usually the statements (i) and (iii) are thought to provide the experimental verifications of the *TCP* theorem. However, the situation is not so simple, as we shall see shortly. We know that masses of π^+ and π^-, or of e^+ and e^-, are equal to a good degree of accuracy. Probably the best example would be the equality of K^0 and $\bar{K}^0$ meson masses. If the masses of K^0 and $\bar{K}^0$ differed by a small magnitude of the order of 10^{-5} eV, then our picture of $K_1{}^0$ and $K_2{}^0$ would be quite different[10] from what it is now. This is because the mass difference[11] between $K_1{}^0$ and $K_2{}^0$ is of the order of 10^{-5} eV. Thus, we may be not far wrong to say that the masses of K^0 and $\bar{K}^0$ (when we neglect the weak interactions) have to be the same within an accuracy of 10^{-5} eV, though we cannot properly formulate this problem in the framework of the present field theory. If we believe in the *TCP* theorem since it holds under very general conditions in the present field theory,[12] then we can easily explain such equivalence of masses of a particle and its antiparticle. However, it is also possible to explain these by postulating the invariance of the theory under *C* or *CP* conjugation while *TCP* is violated (so that *TP* or *T* is violated accordingly). Thus, the equivalence of masses of a particle and its antiparticle does not give direct evidence for the validity of the *TCP* theorem. The same is true for the equality of lifetimes of a particle and its antiparticle, since by the same reason it might be due to *C* or *CP* invariance of the theory while *TCP* might be violated. Therefore, we have to test the *TCP* theorem more directly. We note that Bernstein and Michel[13] have tried to check the validity of the *TCP* theorem in the case of $\pi^0 \to 2\gamma$ decay. What we are proposing in this note is to consider the same problem in the case of the elastic proton-antiproton scattering, Eq. (2), since it is more feasible and will give many possibilities. Of course, in our procedures, we can test separately the *P*, *C*, and *T* invariances of the theory. The suggested experiments are of the same type as those which have been used to determine the nucleon-nucleon scattering matrix and also to test its time-reversal invariance.[14]

II. FORMULATION

Let us consider the scattering matrix elements $M(\alpha\to\beta)$ and $M(\tilde{\beta}\to\tilde{\alpha})$ for the transition $\alpha\to\beta$ and for the antiparticle transition $\tilde{\beta}\to\tilde{\alpha}$, respectively. We neglect the weak interactions to be more definite. Then these transition matrix elements are given by

$$M(\alpha\to\beta)=\langle\beta(\text{out})|\alpha(\text{in})\rangle, \tag{3a}$$

$$M(\tilde{\beta}\to\tilde{\alpha})=\langle\tilde{\alpha}(\text{out})|\tilde{\beta}(\text{in})\rangle. \tag{3b}$$

If we define $\tilde{\alpha}$ and $\tilde{\beta}$ to be the states which can be obtained from α and β by the *TCP* operation, respectively, then the *TCP* theorem implies that we have

$$M(\alpha\to\beta)=M(\tilde{\beta}\to\tilde{\alpha}). \tag{4}$$

Now, setting $\alpha=\beta$ and taking the imaginary part of Eq. (4), we get

$$\sigma_{\text{total}}(\alpha)=\sigma_{\text{total}}(\tilde{\alpha}) \tag{5}$$

due to the well-known optical theorem, where σ_{total} means the total cross section. Equation (5) provides the required proof of the statement (iv) mentioned in the Introduction.

Hereafter, we restrict ourselves to the special case of proton-antiproton scattering, Eq. (2). Let us take $\mathbf{p}$, $(-\mathbf{p})$, and $\mathbf{p}'$, $(-\mathbf{p}')$ to be the momenta of the initial proton (antiproton) and of the final proton (antiproton) in the barycentric system, respectively. Furthermore, we specify the spin operators of the antiproton and of the proton as $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, respectively. By the *C*, *P*, and *T* operations, we have the following transformations:

$$\begin{aligned}
&C:\ \boldsymbol{\sigma}_1\leftrightarrows\boldsymbol{\sigma}_2,\quad \mathbf{p}\to-\mathbf{p},\quad \mathbf{p}'\to-\mathbf{p}';\\
&T:\ \boldsymbol{\sigma}_1\to-\boldsymbol{\sigma}_1,\quad \boldsymbol{\sigma}_2\to-\boldsymbol{\sigma}_2,\quad \mathbf{p}\to-\mathbf{p}',\quad \mathbf{p}'\to-\mathbf{p};\\
&P:\ \boldsymbol{\sigma}_1\to\boldsymbol{\sigma}_1,\quad \boldsymbol{\sigma}_2\to\boldsymbol{\sigma}_2,\quad \mathbf{p}\to-\mathbf{p},\quad \mathbf{p}'\to-\mathbf{p}';
\end{aligned} \tag{6}$$

so that by *TCP*, we have to make the following transformation:

$$TCP:\ \boldsymbol{\sigma}_1\to-\boldsymbol{\sigma}_2,\ \boldsymbol{\sigma}_2\to-\boldsymbol{\sigma}_1,\ \mathbf{p}\to-\mathbf{p}',\ \mathbf{p}'\to-\mathbf{p}. \tag{7}$$

Equation (4) demands that the matrix element for the process Eq. (2) should be invariant under the transformation Eq. (7), if *TCP* is valid.

Now, we list the sixteen invariant quantities for the process Eq. (2) in the following, along with the definition of a complex scalar coefficient for each invariant quantity:

$$\begin{aligned}
&M(0)=a,\\
&M(11)=C(11)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{l},\quad M(12)=C(12)(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{m},\\
&M(13)=C(13)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{l},\quad M(14)=C(14)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{m},\\
&M(15)=c(\boldsymbol{\sigma}_1+\boldsymbol{\sigma}_2)\cdot\mathbf{n},\qquad M(16)=C(16)(\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2)\cdot\mathbf{n},\\
&M(21)=g[(\boldsymbol{\sigma}_1\cdot\mathbf{l})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{l})+(\boldsymbol{\sigma}_1\cdot\mathbf{m})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{m})],\\
&M(22)=C(22)[(\boldsymbol{\sigma}_1\cdot\mathbf{l})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{m})+(\boldsymbol{\sigma}_1\cdot\mathbf{m})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{l})],\\
&M(23)=C(23)[(\boldsymbol{\sigma}_1\cdot\mathbf{l})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{m})-(\boldsymbol{\sigma}_1\cdot\mathbf{m})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{l})],\\
&M(24)=h[(\boldsymbol{\sigma}_1\cdot\mathbf{l})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{l})-(\boldsymbol{\sigma}_1\cdot\mathbf{m})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{m})],\\
&M(25)=C(25)[(\boldsymbol{\sigma}_1\cdot\mathbf{l})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{n})+(\boldsymbol{\sigma}_1\cdot\mathbf{n})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{l})],\\
&M(26)=C(26)[(\boldsymbol{\sigma}_1\cdot\mathbf{m})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{n})+(\boldsymbol{\sigma}_1\cdot\mathbf{n})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{m})],\\
&M(27)=m(\boldsymbol{\sigma}_1\cdot\mathbf{n})\cdot(\boldsymbol{\sigma}_2\cdot\mathbf{n}),\\
&M(28)=C(28)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\mathbf{l},\\
&M(29)=C(29)(\boldsymbol{\sigma}_1\times\boldsymbol{\sigma}_2)\cdot\mathbf{m}.
\end{aligned} \tag{8}$$

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[10] M. L. Good has used a similar argument against a possibility of a negative gravitational mass for K^0 (or $\bar{K}^0$) while $\bar{K}^0$ (or K^0) has a positive gravitational mass: See M. L. Good, Phys. Rev. **121**, 311 (1961).

[11] R. P. Matsen, L. Oswald, W. M. Powell, H. S. White, and O. Piccioni, Phys. Rev. Letters **4**, 418 (1960).

[12] W. Pauli, in *Niels Bohr and the Development of Physics* (Pergamon Press, New York, 1955), p. 30; R. Jost, Helv. Phys. Acta **30**, 409 (1957).

[13] J. Bernstein and L. Michel, Phys. Rev. **118**, 871 (1960).

[14] A. E. Woodruff, Ann. Phys. (New York) **7**, 65 (1959). See also Abashian and Hafner, Hillman, *et al.*, and Hwang, *et al.*, reference 2.

TABLE I. Transformation properties of the matrix elements of Eq. (8) under P, C, T, and TCP. The symbols X and 0 represent the noninvariance and invariance under the corresponding transformations, respectively.

	P	C	T	TCP
$M(0)$	0	0	0	0
$M(11)$	X	X	0	0
$M(12)$	X	X	X	X
$M(13)$	X	0	0	X
$M(14)$	X	0	X	0
$M(15)$	0	0	0	0
$M(16)$	0	X	0	X
$M(21)$	0	0	0	0
$M(22)$	0	0	X	X
$M(23)$	0	X	X	0
$M(24)$	0	0	0	0
$M(25)$	X	X	0	0
$M(26)$	X	X	X	X
$M(27)$	0	0	0	0
$M(28)$	X	0	X	0
$M(29)$	X	0	0	X

Here we have employed the following coordinate system:

$$\mathbf{l}=-(\mathbf{p}+\mathbf{p}')/|\mathbf{p}+\mathbf{p}'|,$$
$$\mathbf{m}=(\mathbf{p}-\mathbf{p}')/|\mathbf{p}-\mathbf{p}'|, \quad (9)$$
$$\mathbf{n}=(\mathbf{p}\times\mathbf{p}')/|\mathbf{p}\times\mathbf{p}'|.$$

We note that occurrence of these matrix elements is similar to that for nucleon-nucleon scattering.[14]

For these matrix elements, we can check their properties under P, C, T, and TCP operations and we list them in Table I, where the symbols X and 0 represent the noninvariance and invariance under the corresponding transformations, respectively. So, if we find the presence of one of $M(12)$, $M(13)$, $M(16)$, $M(22)$ $M(26)$, or $M(29)$, then TCP is violated.

We can follow exactly the same procedure for the process $p+\bar{p}\rightarrow n+\bar{n}$, if we assume charge symmetry.

III. POSSIBLE EXPERIMENTAL TESTS IN $\bar{p}$-p ELASTIC SCATTERING

The experimental quantities in proton-antiproton elastic scattering are exhausted by the following general expression given by Wolfenstein and Ashkin.[15]

$$I_0\langle S^\mu\rangle_f=\sum_\nu \langle S^\nu\rangle_i(1/4)\,\mathrm{Tr}(MS^\nu M^\dagger S^\mu),\qquad \mu,\nu=1,2,\cdots 16. \quad (10)$$

Here S^μ and S^ν are any of the sixteen base vectors for the compound spin space of proton and antiproton spins, M is the scattering matrix, $M^\dagger$ is the Hermitian conjugate of M, I_0 is the differential cross section for the unpolarized scattering, and $\langle S^\mu\rangle_{f,i}$ indicates the expectation value of S^μ in the final (initial) state. In Appendix 1 we list all the possible observable quantities in notations similar to those used by Schumacher and Bethe.[16]

We calculate some of the simpler experimental quantities, neglecting the order of the squares of the "violating" terms (that is, all the terms except a, c, g, h, and m). It is convenient to apply the theorem stated in Appendix 2 for finding out which experimental quantities will possibly reveal existence of a particular violating term: Now let us choose $(\boldsymbol{\sigma}_1\cdot\mathbf{n})(\boldsymbol{\sigma}_2\cdot\mathbf{n})$ as the operator O in Appendix 2. This operator commutes with the partial scattering matrices which conserve parity, and anticommutes with the parity-nonconserving partial scattering matrices. Thus, we expect that the interference terms include only $C(16)$, $C(22)$, and $C(23)$, and no other violating terms in the following experimental quantities:

$$I_0,\quad P_n^{(1),(2)},\quad A_n^{(1),(2)},\quad \mathfrak{D}_{nn}^{(1),(2)},\quad \mathfrak{D}_{mm}^{(1),(2)},\quad \mathfrak{D}_{ll}^{(1),(2)},\quad \mathfrak{D}_{ml}^{(1),(2)},\quad \mathfrak{D}_{lm}^{(1),(2)},$$
$$\mathcal{C}_{nn},\quad \mathcal{C}_{mm},\quad \mathcal{C}_{ll},\quad \mathcal{C}_{lm},\quad \mathcal{C}_{ml},\quad \mathcal{A}_{nn},\quad \mathcal{A}_{mm},\quad \mathcal{A}_{ll},\quad \mathcal{A}_{ml},\quad \mathcal{A}_{lm},$$

and so forth.

All the other components of the quantities consist of the interference terms involving the other eight parity-nonconserving partial scattering matrices.

Moreover, by direct calculations we find that I_0, $\mathcal{C}_{nn}$, $\mathfrak{D}_{nn}^{(1)}$, $\mathcal{K}_{nn}^{(1)}$, and $\mathcal{A}_{nn}$ include no violating terms in the approximation we take. Also, the relations $\mathcal{C}_{nn}=\mathcal{A}_{nn}$ and $\mathcal{A}_{ll}+\mathcal{A}_{mm}=\mathcal{C}_{ll}+\mathcal{C}_{mm}$, which hold when only the five "nonviolating" terms exist, are still valid with inclusion of all the "violating" terms in the above-mentioned approximation.

The expressions for some of the simpler experimental quantities are given in Appendix 3.

Although there are a number of experiments which can, in principle, detect the existence of the M-matrix components which violate any of the T, C, P, and TCP invariances, only those which involve $C(16)$, $C(22)$, $C(23)$, and no others are simple enough to be realistic quantities to be used as such tests: All the experimental quantities to which the parity nonconservation could give nonvanishing values are very much more complicated, depending on eight violating M-matrix terms. Therefore, we describe some of the simpler experiments here which may reveal the existence of $C(16)$, $C(22)$, and $C(23)$ terms. According to the occurrence of a particular combination of these terms, we can have the following three types of experiments:

(I) Detection of $C(23)$ and $C(16)$: test for C or C and T violation.

(II) Detection of $C(23)$ and $C(22)$: test for T or C and T violation.

(III) Detection of $C(22)$ and/or $C(16)$: test for TCP violation through C and/or T violation.

[15] L. Wolfenstein and J. Ashkin, Phys. Rev. **85**, 947 (1952).

[16] C. R. Schumacher and H. A. Bethe, Phys. Rev. **121**, 1534 (1961).

Type (I)

(a) Comparison of the polarizations of $\bar{p}$ and p in the $\bar{p}$-p unpolarized scattering:

$$I_0(P_n^{(1)}-P_n^{(2)})=4\,\mathrm{Re}[(a-m)C^*(16)]+8\,\mathrm{Im}[gC^*(23)]. \quad \text{(Ia)}$$

If there exists a nonvanishing difference, the charge-conjugation invariance is violated with or without TCP violation.

(b) Comparison of the asymmetries of $\bar{p}$ and p produced by the scatterings with polarized $\bar{p}$ incident on an unpolarized proton target and with unpolarized $\bar{p}$ on a polarized proton target, respectively:

$$I_0(A_n^{(1)}-A_n^{(2)})=4\,\mathrm{Re}[(a-m)C^*(16)]-8\,\mathrm{Im}[gC^*(23)]. \quad \text{(Ib)}$$

The combination of Eqs. (Ia) and (Ib) enables us to decide whether or not the TCP theorem is violated through charge-conjugation violation.

The following quantities also give similar information:

$$I_0(\mathcal{C}_{lm}-\mathcal{C}_{ml})=4\,\mathrm{Re}[(a-m)C^*(23)]-8\,\mathrm{Im}[gC^*(16)], \quad \text{(Ic)}$$

$$I_0(\mathcal{A}_{lm}-\mathcal{A}_{ml})=4\,\mathrm{Re}[(a-m)C^*(23)]+8\,\mathrm{Im}[gC^*(16)], \quad \text{(Id)}$$

$$I_0(\mathcal{K}_{lm}^{(1)}-\mathcal{K}_{ml}^{(1)})=4\,\mathrm{Re}[(a+m)C^*(23)]-8\,\mathrm{Im}[hC^*(16)]. \quad \text{(Ie)}$$

Type (II)

(a) Comparison between the asymmetry of $\bar{p}$ produced in the scattering with polarized $\bar{p}$ incident on an unpolarized target and the polarization of $\bar{p}$ in $\bar{p}$-p unpolarized scattering[17]:

$$I_0(P_n^{(1)}-A_n^{(1)})=8\,\mathrm{Im}[gC^*(23)]-8\,\mathrm{Im}[hC^*(22)]. \quad \text{(IIa)}$$

(b) Comparison between the asymmetry of the recoil protons in the scattering with unpolarized $\bar{p}$ incident on a polarized proton target and the polarization of the recoil protons produced in $\bar{p}$-p unpolarized scattering:

$$I_0(P_n^{(2)}-A_n^{(2)})=-8\,\mathrm{Im}[gC^*(23)]-8\,\mathrm{Im}[hC^*(22)]. \quad \text{(IIb)}$$

Equation (IIa) or Eq. (IIb) can be a test for a possible breakdown of time-reversal invariance with or without TCP invariance. The combination of these two will test whether or not the TCP theorem is violated through time-reversal breakdown.

[17] In connection with T violation, A. E. Woodruff (reference 14) gives similar relations between P and A and between $\mathcal{C}_{ij}$ and $\mathcal{A}_{ij}$ in the case of proton-proton scattering.

The following more complicated experimental quantities also can give the same information as the above:

$$I_0(\mathcal{D}_{ml}^{(1)}+\mathcal{D}_{lm}^{(1)})=8\,\mathrm{Re}[gC^*(22)]-8\,\mathrm{Re}[hC^*(23)], \quad \text{(IIc)}$$

$$I_0(\mathcal{A}_{lm}+\mathcal{C}_{lm})=4\,\mathrm{Re}[(a+m)C^*(22)]+4\,\mathrm{Re}[(a-m)C^*(23)], \quad \text{(IId)}$$

$$I_0(\mathcal{A}_{ml}+\mathcal{C}_{ml})=4\,\mathrm{Re}[(a+m)C^*(22)]-4\,\mathrm{Re}[(a-m)C^*(23)]. \quad \text{(IIe)}$$

We note that Eq. (IIc) is a generalization of the familiar relationship[14,18] among the triple-scattering parameters A, R, A', R' and the center-of-mass scattering angle θ of antiprotons:

$$(A+R')\cos(\theta/2)-(A'-R)\sin(\theta/2)=8\,\mathrm{Re}[gC^*(22)]-8\,\mathrm{Re}[hC^*(23)]. \quad \text{(IIc')}$$

Type (III)

There is no combination of two simplest experimental quantities for the test of this category. We give a few of the least complicated quantities in the following:

$$I_0(\mathcal{A}_{mm}-\mathcal{C}_{mm})=8\,\mathrm{Im}[cC^*(22)], \quad \text{(IIIa)}$$

$$I_0(\mathcal{A}_{ll}-\mathcal{C}_{ll})=-8\,\mathrm{Im}[cC^*(22)], \quad \text{(IIIb)}$$

$$I_0(\mathcal{K}_{lm}^{(1)}+\mathcal{K}_{ml}^{(1)})=4\,\mathrm{Re}[(a-m)C^*(22)]-8\,\mathrm{Im}[hC^*(16)]. \quad \text{(IIIc)}$$

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APPENDIX 1

Unpolarized differential cross section:

$$I_0=\tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger). \quad \text{(A1.1)}$$

Polarization vectors:

$$I_0\mathbf{P}^{(1)}=\tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger\boldsymbol{\sigma}_1),\quad I_0\mathbf{P}^{(2)}=\tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger\boldsymbol{\sigma}_2). \quad \text{(A1.2)}$$

Asymmetry vectors:

$$I_0\mathbf{A}^{(1)}=\tfrac{1}{4}\,\mathrm{Tr}(M^\dagger M\boldsymbol{\sigma}_1),\quad I_0\mathbf{A}^{(2)}=\tfrac{1}{4}\,\mathrm{Tr}(M^\dagger M\boldsymbol{\sigma}_2). \quad \text{(A1.3)}$$

Depolarization tensors:

$$I_0\mathcal{D}_{ij}^{(1)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{1i}M^\dagger\sigma_{1j}),\quad I_0\mathcal{D}_{ij}^{(2)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{2i}M^\dagger\sigma_{2j}). \quad \text{(A1.4)}$$

Polarization transfer tensors:

[18] L. Wolfenstein, Ann. Rev. Nuclear Sci. 6, 43 (1956).

$$I_0\mathcal{K}_{ij}^{(1)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{1i}M^\dagger\sigma_{2j}),$$
$$I_0\mathcal{K}_{ij}^{(2)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{2i}M^\dagger\sigma_{1j}). \tag{A1.5}$$

Correlation tensor:

$$I_0\mathcal{C}_{ij}=\tfrac{1}{4}\,\mathrm{Tr}(MM^\dagger\sigma_{1i}\sigma_{2j}). \tag{A1.6}$$

Asymmetry tensor:

$$I_0\mathcal{A}_{ij}=\tfrac{1}{4}\,\mathrm{Tr}(M^\dagger M\sigma_{1i}\sigma_{2j}). \tag{A1.7}$$

Correlation tensors for the polarized scattering:

$$I_0\mathcal{C}_{i,rs}^{(1)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{1i}M^\dagger\sigma_{1r}\sigma_{2s}),$$
$$I_0\mathcal{C}_{i,rs}^{(2)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{2i}M^\dagger\sigma_{1r}\sigma_{2s}), \tag{A1.8}$$
$$I_0\mathcal{C}_{ij,rs}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{1i}\sigma_{2j}M^\dagger\sigma_{1r}\sigma_{2s}).$$

Polarization transfer tensors with a polarized target:

$$I_0\mathcal{P}_{ij,r}^{(1)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{1i}\sigma_{2j}M^\dagger\sigma_{1r}),$$
$$I_0\mathcal{P}_{ij,r}^{(2)}=\tfrac{1}{4}\,\mathrm{Tr}(M\sigma_{1i}\sigma_{2j}M^\dagger\sigma_{2r}). \tag{A1.9}$$

APPENDIX 2[19]

Suppose there exists a unitary operator O which transforms $M(i)$ and S^μ as follows:

$$OM(i)O^{-1}=\pm M(i), \tag{A2.1}$$
$$OS^\mu O^{-1}=\pm S^\mu. \tag{A2.2}$$

We write the total M matrix as $M=M_e+M_o$, where M_e and M_o transform as $OM_eO^{-1}=+M_e$, $OM_oO^{-1}=-M_o$. Then, the quantity $A=\mathrm{Tr}(MS^\nu M^\dagger S^\mu)$ can be expressed as

$$(1)\quad A=\mathrm{Tr}(M_eS^\nu M_e{}^\dagger S^\mu+M_oS^\nu M_o{}^\dagger S^\mu),$$
or (A2.3)
$$(2)\quad A=\mathrm{Tr}(M_eS^\nu M_o{}^\dagger S^\mu+M_oS^\nu M_e{}^\dagger S^\mu),$$

according to (1) if S^ν and S^μ behave similarly under the transformation, or (2) if they transform differently.

APPENDIX 3

$$I_0=|a|^2+2|c|^2+|m|^2+2|g|^2+2|h|^2. \tag{A3.1}$$

$I_0\mathbf{P}_1$:

$$\begin{aligned}
I_0\langle\boldsymbol{\sigma}_1\rangle_f\cdot\mathbf{l}=&\,2\,\mathrm{Re}[(a+g+h)C^*(11)+cC^*(25)]+2\,\mathrm{Im}[(m-g+h)C^*(26)+cC^*(12)]\\
&-2\,\mathrm{Re}[(-a+g+h)C^*(13)+cC^*(29)]+2\,\mathrm{Im}[(m+g-h)C^*(28)+cC^*(14)],\\
I_0\langle\boldsymbol{\sigma}_1\rangle_f\cdot\mathbf{m}=&\,2\,\mathrm{Re}[(a+g-h)C^*(12)+cC^*(26)]+2\,\mathrm{Im}[(-m+g+h)C^*(25)-cC^*(11)]\\
&+2\,\mathrm{Re}[(a-g+h)C^*(14)+cC^*(28)]+2\,\mathrm{Im}[(m+g+h)C^*(29)-cC^*(13)],\\
I_0\langle\boldsymbol{\sigma}_1\rangle_f\cdot\mathbf{n}=&\,2\,\mathrm{Re}[(a+m)c^*+(a-m)C^*(16)]+4\,\mathrm{Im}[gC^*(23)-hC^*(22)].
\end{aligned} \tag{A3.2}$$

$I_0\mathbf{P}_2$:

$$\begin{aligned}
I_0\langle\boldsymbol{\sigma}_2\rangle_f\cdot\mathbf{l}=&\,2\,\mathrm{Re}[(a+g+h)C^*(11)+cC^*(25)]+2\,\mathrm{Im}[(m-g+h)C^*(26)+cC^*(12)]\\
&+2\,\mathrm{Re}[(-a+g+h)C^*(13)+cC^*(29)]-2\,\mathrm{Im}[(m+g-h)C^*(28)+cC^*(14)],\\
I_0\langle\boldsymbol{\sigma}_2\rangle_f\cdot\mathbf{m}=&\,2\,\mathrm{Re}[(a+g-h)C^*(12)+cC^*(26)]+2\,\mathrm{Im}[(-m+g+h)C^*(25)-cC^*(11)]\\
&-2\,\mathrm{Re}[(a-g+h)C^*(14)+cC^*(18)]-2\,\mathrm{Im}[(m+g+h)C^*(29)-cC^*(13)],\\
I_0\langle\boldsymbol{\sigma}_2\rangle_f\cdot\mathbf{n}=&\,2\,\mathrm{Re}[(a+m)c^*-(a-m)C^*(16)]-4\,\mathrm{Im}[gC^*(23)+hC^*(22)].
\end{aligned} \tag{A3.3}$$

$I_0\mathbf{A}^{(1)}$:

$$\begin{aligned}
I_0\mathbf{A}^{(1)}\mathbf{l}=&\,2\,\mathrm{Re}[(a+g+h)C^*(11)+cC^*(25)]-2\,\mathrm{Im}[(m-g+h)C^*(26)+cC^*(12)]\\
&-2\,\mathrm{Re}[(-a+g+h)C^*(13)+cC^*(29)]-2\,\mathrm{Im}[(m+g-h)C^*(28)+cC^*(14)];\\
I_0\mathbf{A}^{(1)}\mathbf{m}=&\,2\,\mathrm{Re}[(a+g-h)C^*(12)+cC^*(26)]-2\,\mathrm{Im}[(-m+g+h)C^*(25)-cC^*(11)]\\
&+2\,\mathrm{Re}[(a-g+h)C^*(14)+cC^*(28)]-2\,\mathrm{Im}[(m+g+h)C^*(29)-cC^*(13)],\\
I_0\mathbf{A}^{(1)}\mathbf{n}=&\,2\,\mathrm{Re}[(a+m)c^*+(a-m)C^*(16)]-4\,\mathrm{Im}[gC^*(23)-hC^*(22)].
\end{aligned} \tag{A3.4}$$

$I_0\mathbf{A}^{(2)}$:

$$\begin{aligned}
I_0\mathbf{A}^{(2)}\mathbf{l}=&\,2\,\mathrm{Re}[(a+g+h)C^*(11)+cC^*(25)]-2\,\mathrm{Im}[(m-g+h)C^*(26)+cC^*(12)]\\
&+2\,\mathrm{Re}[(-a+g+h)C^*(13)+cC^*(29)]+2\,\mathrm{Im}[(m-g-h)C^*(28)+cC^*(14)],\\
I_0\mathbf{A}^{(2)}\mathbf{m}=&\,2\,\mathrm{Re}[(a+g-h)C^*(12)+cC^*(26)]-2\,\mathrm{Im}[(-m+g+h)C^*(25)-cC^*(11)]\\
&-2\,\mathrm{Re}[(a-g+h)C^*(14)+cC^*(28)]+2\,\mathrm{Im}[(m+g+h)C^*(29)-cC^*(13)],\\
I_0\mathbf{A}^{(2)}\mathbf{n}=&\,2\,\mathrm{Re}[(a+m)c^*-(a-m)C^*(16)]+4\,\mathrm{Im}[gC^*(23)+hC^*(22)].
\end{aligned} \tag{A3.5}$$

[19] This is a generalization of the argument used by J. L. Gammel and R. M. Thaler in connection with parity conservation in nucleon-nucleon scattering. See *Progress in Elementary Particle and Cosmic-Ray Physics* (North-Holland Publishing Company, Amsterdam, 1960), Vol. V, p. 97.

$I_0\mathfrak{C}_{ij}$:

$$I_0\mathfrak{C}_{nn}=2\,\mathrm{Re}(ma^*)+2|c|^2+2|h|^2-2|g|^2,$$

$$I_0\mathfrak{C}_{mm}=2\,\mathrm{Re}[a(g^*-h^*)-m(g^*+h^*)]-4\,\mathrm{Im}[cC^*(22)],$$

$$I_0\mathfrak{C}_{ll}=2\,\mathrm{Re}[a(g^*+h^*)-m(g^*-h^*)]+4\,\mathrm{Im}[cC^*(22)],$$

$$I_0\mathfrak{C}_{nl}=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]+2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)] +2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]-2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)],$$

$$I_0\mathfrak{C}_{ln}=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]+2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)] -2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]+2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)], \quad \text{(A3.6)}$$

$$I_0\mathfrak{C}_{lm}=-4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)+(a-m)C^*(23)]-4\,\mathrm{Im}[gC^*(16)],$$

$$I_0\mathfrak{C}_{ml}=-4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)-(a-m)C^*(23)]+4\,\mathrm{Im}[gC^*(16)],$$

$$I_0\mathfrak{C}_{mn}=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]-2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)] +2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]-2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)],$$

$$I_0\mathfrak{C}_{nm}=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]-2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)] -2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]+2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)].$$

$I_0\mathfrak{D}_{ij}^{(1)}$:

$$I_0\mathfrak{D}_{nn}^{(1)}=|a|^2+2|c|^2+|m|^2-2|g|^2-2|h|^2,$$

$$I_0\mathfrak{D}_{mm}^{(1)}=|a|^2-|m|^2-4\,\mathrm{Re}(gh^*)-4\,\mathrm{Re}[cC^*(16)],$$

$$I_0\mathfrak{D}_{ll}^{(1)}=|a|^2-|m|^2+4\,\mathrm{Re}(gh^*)-4\,\mathrm{Re}[cC^*(16)],$$

$$I_0\mathfrak{D}_{nl}^{(1)}=2\,\mathrm{Re}[(-m+g+h)C^*(29)+(m+g+h)C^*(25)+(C(13)+C(11))c^*] +2\,\mathrm{Im}[(a-g+h)C^*(12)+(a+g-h)C^*(14)+c(C^*(26)+C^*(28))],$$

$$I_0\mathfrak{D}_{ln}^{(1)}=2\,\mathrm{Re}[(-m+g+h)C^*(29)+(m+g+h)C^*(25)+c^*(C(13)+C(11))] -2\,\mathrm{Im}[(a-g+h)C^*(12)+(a+g-h)C^*(14)+c(C^*(26)+C^*(28))], \quad \text{(A3.7)}$$

$$I_0\mathfrak{D}_{nm}^{(1)}=2\,\mathrm{Re}[(m+g-h)C^*(26)+(m-g+h)C^*(28)+c(C^*(12)+C^*(14))] -2\,\mathrm{Im}[(a+g+h)C^*(13)+(a-g-h)C^*(11)+c(C^*(25)-C^*(29))],$$

$$I_0\mathfrak{D}_{mn}^{(1)}=2\,\mathrm{Re}[(m+g-h)C^*(26)+(m-g+h)C^*(28)+c(C^*(12)+C^*(14))] +2\,\mathrm{Im}[(a+g+h)C^*(13)+(a-g-h)C^*(11)+c(C^*(25)-C^*(29))],$$

$$I_0\mathfrak{D}_{ml}^{(1)}=-2\,\mathrm{Im}[(a-m)c^*+(a+m)C^*(16)]+4\,\mathrm{Re}[gC^*(22)-hC^*(23)],$$

$$I_0\mathfrak{D}_{lm}^{(1)}=+2\,\mathrm{Im}[(a-m)c^*+(a+m)C^*(16)]+4\,\mathrm{Re}[gC^*(22)-hC^*(23)].$$

We note that the components of the depolarization tensor $\mathfrak{D}_{ij}^{(1)}$ are related linearly to the familiar triple-scattering parameters D, A, R, A', and R' defined for nucleon-nucleon scattering[15,18]:

$$\begin{aligned}
\mathfrak{D}_{nn}^{(1)}&=D,\\
\mathfrak{D}_{mm}^{(1)}\cos(\theta/2)+\mathfrak{D}_{lm}^{(1)}\sin(\theta/2)&=R,\\
\mathfrak{D}_{lm}^{(1)}\cos(\theta/2)-\mathfrak{D}_{mm}^{(1)}\sin(\theta/2)&=A,\\
\mathfrak{D}_{ml}^{(1)}\cos(\theta/2)+\mathfrak{D}_{ll}^{(1)}\sin(\theta/2)&=R',\\
\mathfrak{D}_{ll}^{(1)}\cos(\theta/2)-\mathfrak{D}_{ml}^{(1)}\sin(\theta/2)&=A',
\end{aligned} \quad \text{(A3.7')}$$

where θ is the center-of-mass scattering angle of antiprotons.

$I_0\mathcal{A}_{ij}$:

$$
\begin{aligned}
I_0\mathcal{A}_{nn}&=2\,\mathrm{Re}(ma^*)+2|c|^2+2|h|^2-2|g|^2,\\
I_0\mathcal{A}_{mm}&=2\,\mathrm{Re}[a(g^*-h^*)-m(g^*+h^*)]+4\,\mathrm{Im}[cC^*(22)],\\
I_0\mathcal{A}_{ll}&=2\,\mathrm{Re}[a(g^*+h^*)-m(g^*-h^*)]-4\,\mathrm{Im}[cC^*(22)],\\
I_0\mathcal{A}_{nl}&=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]-2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)]\\
&\qquad+2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]+2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)],\\
I_0\mathcal{A}_{ln}&=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]-2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)]\\
&\qquad-2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]-2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)],\\
I_0\mathcal{A}_{lm}&=+4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)+(a-m)C^*(23)]+4\,\mathrm{Im}[gC^*(16)],\\
I_0\mathcal{A}_{ml}&=+4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)-(a-m)C^*(23)]-4\,\mathrm{Im}[gC^*(16)],\\
I_0\mathcal{A}_{mn}&=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]+2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)]\\
&\qquad+2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]+2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)],\\
I_0\mathcal{A}_{nm}&=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]+2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)]\\
&\qquad-2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]-2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)].
\end{aligned}
\tag{A3.8}
$$

$I_0\mathcal{K}_{ij}^{(1)}$:

$$
\begin{aligned}
I_0\mathcal{K}_{nn}^{(1)}&=2\,\mathrm{Re}(am^*)+2|c|^2+2|g|^2-2|h|^2,\\
I_0\mathcal{K}_{mm}^{(1)}&=2\,\mathrm{Re}[(a+m)g^*+(a-m)h^*]+4\,\mathrm{Im}[cC^*(23)],\\
I_0\mathcal{K}_{ll}^{(1)}&=2\,\mathrm{Re}[(a+m)g^*-(a-m)h^*]+4\,\mathrm{Im}[cC^*(23)],\\
I_0\mathcal{K}_{nl}^{(1)}&=2\,\mathrm{Re}[(a-g+h)C^*(25)+(a+g-h)C^*(29)-cC^*(13)+cC^*(11)]\\
&\qquad+2\,\mathrm{Im}[(g+h-m)C^*(14)+(m+g+h)C^*(12)+cC^*(26)+cC^*(28)],\\
I_0\mathcal{K}_{ln}^{(1)}&=2\,\mathrm{Re}[(a-g+h)C^*(25)-(a+g-h)C^*(29)+cC^*(13)+cC^*(11)]\\
&\qquad+2\,\mathrm{Im}[(g+h-m)C^*(14)-(m+g+h)C^*(12)-cC^*(26)-cC^*(28)],\\
I_0\mathcal{K}_{nm}^{(1)}&=2\,\mathrm{Re}[(a-g-h)C^*(26)-(a+g+h)C^*(28)+cC^*(12)-cC^*(14)]\\
&\qquad+2\,\mathrm{Im}[(m+h-g)C^*(13)-(m+g-h)C^*(11)-cC^*(29)-cC^*(25)],\\
I_0\mathcal{K}_{mn}^{(1)}&=2\,\mathrm{Re}[(a-g-h)C^*(26)+(a+g+h)C^*(28)+cC^*(12)+cC^*(14)]\\
&\qquad+2\,\mathrm{Im}[(m+h-g)C^*(13)+(m+g-h)C^*(11)-cC^*(29)+cC^*(25)],\\
I_0\mathcal{K}_{lm}^{(1)}&=2\,\mathrm{Re}[(a-m)C^*(22)+(a+m)C^*(23)]+4\,\mathrm{Im}[gc^*-hC^*(16)],\\
I_0\mathcal{K}_{ml}^{(1)}&=2\,\mathrm{Re}[(a-m)C^*(22)-(a+m)C^*(23)]-4\,\mathrm{Im}[gc^*+hC^*(16)].
\end{aligned}
\tag{A3.9}
$$

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Width of Three-Pion Resonances*

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A model of three-pion resonances is constructed taking into account only π-ρ intermediate states. The π-ρ interaction is replaced by a simple pole. The calculated ω-resonance width is approximately 10–20 MeV, depending on the range of the π-ρ interaction. These values of the width are consistent with present experimental data.

WE have estimated the width of $I=0$ three-pion resonances on the basis of a disperion-theoretic calculation of the three-pion scattering amplitude, finding for the ω meson a full width of 10–20 MeV. We have employed an angular-momentum expansion of the three-pion state in order to reduce the calculation to manageable dimensions. The picture of an $I=0$, three-pion resonance as a quasi-bound state of a pion and a ρ meson then arises from the approximation of consider-

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