$I_0\mathcal{A}_{ij}$:

$$
\begin{aligned}
I_0\mathcal{A}_{nn}&=2\,\mathrm{Re}(ma^*)+2|c|^2+2|h|^2-2|g|^2,\\
I_0\mathcal{A}_{mm}&=2\,\mathrm{Re}[a(g^*-h^*)-m(g^*+h^*)]+4\,\mathrm{Im}[cC^*(22)],\\
I_0\mathcal{A}_{ll}&=2\,\mathrm{Re}[a(g^*+h^*)-m(g^*-h^*)]-4\,\mathrm{Im}[cC^*(22)],\\
I_0\mathcal{A}_{nl}&=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]-2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)]\\
&\quad+2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]+2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)],\\
I_0\mathcal{A}_{ln}&=2\,\mathrm{Re}[(a-h+g)C^*(25)+cC^*(11)]-2\,\mathrm{Im}[(m-g-h)C^*(12)+cC^*(26)]\\
&\quad-2\,\mathrm{Re}[(a+h-g)C^*(29)-cC^*(13)]-2\,\mathrm{Im}[(m+g+h)C^*(14)+cC^*(28)],\\
I_0\mathcal{A}_{lm}&=+4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)+(a-m)C^*(23)]+4\,\mathrm{Im}[gC^*(16)],\\
I_0\mathcal{A}_{ml}&=+4\,\mathrm{Im}(ch^*)+2\,\mathrm{Re}[(a+m)C^*(22)-(a-m)C^*(23)]-4\,\mathrm{Im}[gC^*(16)],\\
I_0\mathcal{A}_{mn}&=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]+2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)]\\
&\quad+2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]+2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)],\\
I_0\mathcal{A}_{nm}&=2\,\mathrm{Re}[(a+g+h)C^*(26)+cC^*(12)]+2\,\mathrm{Im}[(m-g+h)C^*(11)+cC^*(25)]\\
&\quad-2\,\mathrm{Re}[(a-g-h)C^*(28)+cC^*(14)]-2\,\mathrm{Im}[(m+g-h)C^*(13)-cC^*(29)].
\end{aligned}
\tag{A3.8}
$$

$I_0\mathcal{K}_{ij}^{(1)}$:

$$
\begin{aligned}
I_0\mathcal{K}_{nn}^{(1)}&=2\,\mathrm{Re}(am^*)+2|c|^2+2|g|^2-2|h|^2,\\
I_0\mathcal{K}_{mm}^{(1)}&=2\,\mathrm{Re}[(a+m)g^*+(a-m)h^*]+4\,\mathrm{Im}[cC^*(23)],\\
I_0\mathcal{K}_{ll}^{(1)}&=2\,\mathrm{Re}[(a+m)g^*-(a-m)h^*]+4\,\mathrm{Im}[cC^*(23)],\\
I_0\mathcal{K}_{nl}^{(1)}&=2\,\mathrm{Re}[(a-g+h)C^*(25)+(a+g-h)C^*(29)-cC^*(13)+cC^*(11)]\\
&\quad+2\,\mathrm{Im}[(g+h-m)C^*(14)+(m+g+h)C^*(12)+cC^*(26)+cC^*(28)],\\
I_0\mathcal{K}_{ln}^{(1)}&=2\,\mathrm{Re}[(a-g+h)C^*(25)-(a+g-h)C^*(29)+cC^*(13)+cC^*(11)]\\
&\quad+2\,\mathrm{Im}[(g+h-m)C^*(14)-(m+g+h)C^*(12)-cC^*(26)-cC^*(28)],\\
I_0\mathcal{K}_{nm}^{(1)}&=2\,\mathrm{Re}[(a-g-h)C^*(26)-(a+g+h)C^*(28)+cC^*(12)-cC^*(14)]\\
&\quad+2\,\mathrm{Im}[(m+h-g)C^*(13)-(m+g-h)C^*(11)-cC^*(29)-cC^*(25)],\\
I_0\mathcal{K}_{mn}^{(1)}&=2\,\mathrm{Re}[(a-g-h)C^*(26)+(a+g+h)C^*(28)+cC^*(12)+cC^*(14)]\\
&\quad+2\,\mathrm{Im}[(m+h-g)C^*(13)+(m+g-h)C^*(11)-cC^*(29)+cC^*(25)],\\
I_0\mathcal{K}_{lm}^{(1)}&=2\,\mathrm{Re}[(a-m)C^*(22)+(a+m)C^*(23)]+4\,\mathrm{Im}[gc^*-hC^*(16)],\\
I_0\mathcal{K}_{ml}^{(1)}&=2\,\mathrm{Re}[(a-m)C^*(22)-(a+m)C^*(23)]-4\,\mathrm{Im}[gc^*+hC^*(16)].
\end{aligned}
\tag{A3.9}
$$

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# Width of Three-Pion Resonances*

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A model of three-pion resonances is constructed taking into account only $\pi$-$\rho$ intermediate states. The $\pi$-$\rho$ interaction is replaced by a simple pole. The calculated $\omega$-resonance width is approximately 10–20 MeV, depending on the range of the $\pi$-$\rho$ interaction. These values of the width are consistent with present experimental data.

We have estimated the width of $I=0$ three-pion resonances on the basis of a disperion-theoretic calculation of the three-pion scattering amplitude, finding for the $\omega$ meson a full width of 10–20 MeV. We have employed an angular-momentum expansion of the three-pion state in order to reduce the calculation to manageable dimensions. The picture of an $I=0$, three-pion resonance as a quasi-bound state of a pion and a $\rho$ meson then arises from the approximation of consider-

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ing only states of low angular momentum. Such a picture leads to a three-pion resonance whose width is much smaller than the $\rho$-meson width. However, our calculation gives an $\omega$ width of an order of magnitude broader than the estimation of Gell-Mann, Sharp, and Wagner,[1] and several orders of magnitude broader than that of Feinberg.[2]

Let us now consider our quasi-bound-state model in more detail. The angular momentum expansion for three relativistic particles has been worked out by Wick,[3] who generalized the scheme introduced by Dalitz in the study of $\tau$-meson decay.[4] First, two of the three pions, having four-momenta $q_1$ and $q_2$, are characterized by their angular momentum $l$, invariant mass-squared $\sigma$, and helicity $\lambda$; then the two pions are combined with the third (four-momentum $q_3$) to form a state of total angular momentum. Cook and Lee[5] have shown that the generalized unitarity condition[6,7] then simplifies to the form, for $(3m_\pi)^2 \leq s \leq (5m_\pi)^2$,

$$\begin{aligned}&\operatorname{disc}\langle\lambda'l'\sigma'|T^J(s)|\lambda l\sigma\rangle\\&=\frac{2i}{\pi}\sum_{l'',\lambda''}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2}d\sigma''\frac{q(s,\sigma'')}{s^{1/2}}\left(\frac{\sigma''-4}{\sigma''}\right)^{1/2}\\&\quad\times\langle\lambda'l'\sigma'|T^J(s_+)|\lambda''l''\sigma_+''\rangle\\&\quad\times\langle\lambda''l''\sigma_-''|T^J(s_-)|\lambda l\sigma\rangle,\end{aligned}\tag{1}$$

where $s$ is the square of the invariant mass of the three-pion system, and where $q(s,\sigma)$ is the momentum of the third pion in the over-all c.m. system. We use the subscripts $\pm$ as an abbreviation of $\pm i\epsilon$, and use the symbol "disc" to indicate the discontinuity across the branch cut in $s$. Since an $I=0$ state of three pions is totally antisymmetric, only odd values of $l$ occur. We consider only $l=1$, thus neglecting $F$ and higher waves in the pion-pion system. The amplitude must, of course, be symmetrized, but let us defer this complication for a moment. Equation (1) can be simplified still further by considering states of given parity. Specifically, we consider the two quantum-number assignments which have been discussed in connection with three-pion resonances of negative $G$ parity, namely, $1^-$ and $0^-$. Since both these assignments have negative parity, both require $L=1$, where $L$ is the angular momentum of the third pion in the over-all c.m. system. In both cases the scattering amplitude satisfies a unitarity relation of the form of Eq. (1), but as long as states with $l\geq 3$ are neglected, the sum over helicities is not present as long as one considers amplitudes corresponding to $1^-$ and $0^-$. This equation is very similar to the ordinary two-body unitarity condition, except for the additional dependence on $\sigma$.

To facilitate consideration of the dependence on $\sigma$ and $\sigma'$, we factor out the initial and final-state interactions and define a new amplitude $M$ as follows:

$$T^J(s,\sigma',\sigma)=f(\sigma')M^J(s,\sigma',\sigma)f(\sigma),\tag{2}$$

where $f(\sigma)$ is a function having the phase of pion-pion $P$-wave scattering and incorporating the threshold factors appropriate to the present problem. We use a Breit-Wigner type formula,

$$f(\sigma)=\frac{[\gamma(\sigma-4)q^2(s,\sigma)]^{1/2}}{m_\rho^2-\sigma-i\gamma[(\sigma-4)^3/\sigma]^{1/2}}.\tag{3}$$

From the dispersion-theoretic viewpoint the definition of the function $M(s,\sigma',\sigma)$ has been chosen such that the branch points at $\sigma=4m_\pi^2$ and $\sigma'=4m_\pi^2$ associated with pion-pion scattering have been removed. Therefore, we conjecture that $M(s,\sigma',\sigma)$ will not vary rapidly with $\sigma'$ and $\sigma$. In particular, we use the sharply peaked nature of $f(\sigma)$ to simplify Eq. (1) to the form

$$\begin{aligned}&(1/2i)\operatorname{disc}M^J(s,\sigma',\sigma)\\&\qquad\approx\kappa(s)M^J(s_+,\sigma',m_\rho^2)M^J(s_-,\sigma,m_\rho^2),\end{aligned}\tag{4}$$

where

$$\begin{aligned}\kappa(s)=\frac{1}{\pi}\int_{4m_\pi^2}^{(s^{1/2}-m_\pi)^2}d\sigma\frac{q^3(s,\sigma)}{s^{1/2}}\left[\frac{(\sigma-4)^3}{\sigma}\right]^{1/2}\\\times\frac{\gamma}{(m_\rho^2-\sigma)^2+\gamma^2(\sigma-4)^3/\sigma}.\end{aligned}\tag{5}$$

Our procedure up to this point has been parallel to that used by Ball, Frazer, and Nauenberg in treating the state $\pi+\pi+N$.[7] The approximation made in deriving Eq. (4) should be quite reasonable for values of $s$ sufficiently large that the phase-space integration runs over the region of the $\rho$-meson peak. Then we need only assume that $M$ does not vary significantly as a function of $\sigma$ in the region of this peak. In using Eq. (4) for values of $s$ less than $(m_\rho+m_\pi)^2$, we are neglecting the $\sigma$ dependence of $M$ over a wider region.

Note that Eq. (4) is identical in form to the partial-wave unitarity condition for two-body scattering. The properties of the unstable $\rho$ meson are contained in the generalized phase-space factor $\kappa(s)$. In the limit $\gamma\to 0$, $\kappa(s)$ reduces to the two-body $P$-wave phase space $q^3(s,m_\rho^2)/s^{1/2}$.

We shall show later that the effect of symmetrization is to modify the function $\kappa(s)$. Before taking up this question, let us show how the formalism we have developed can be used to estimate the width of three-pion resonances. In order to do this, one must somehow evaluate the interaction between the pions, then solve the $N/D$ equations. The resonance should appear as a

[1] M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters **8**, 261 (1962).
[2] G. Feinberg, Phys. Rev. Letters **8**, 151 (1962).
[3] G. C. Wick, Ann. Phys. (New York) **18**, 65 (1962).
[4] R. H. Dalitz, Phys. Rev. **94**, 1046 (1954).
[5] L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 283 (1962).
[6] R. Blankenbecler, Phys. Rev. **122**, 983 (1960).
[7] J. S. Ball, W. R. Frazer, and M. Nauenberg, University of California, La Jolla, 1962 (to be published).

zero of the $D$ function. As a first rough attempt in this direction we have represented the interaction by a pole (i.e., we have used an effective-range formula) whose residue is adjusted to fit the position of the resonance. We then find that

$$M(s)=\frac{1/\bar{\kappa}(s)}{s_r-s-i[\kappa(s)/\bar{\kappa}(s)]\theta(s-9m_\pi^2)}, \qquad (6)$$

where

$$\bar{\kappa}(s)\equiv\frac{s-s_0}{\pi}\mathcal{P}\int_{(3m_\pi)^2}^{\infty} ds'\frac{\kappa(s')}{(s'-s)(s'-s_0)(s'-s_r)}, \qquad (7)$$

where $s_r$ is the position of the resonance and $s_0$ is the position of the interaction pole. For a narrow resonance we then have a width of $\Gamma\approx\kappa(s_r)/\bar{\kappa}(s_r)$. We shall see below that the width is not very sensitive to the position of $s_0$. In fact, for large negative $s_0$ the dependence is only logarithmic.

Since the width $\Gamma$ is proportional to $\kappa(s_r)$, we can see from Eq. (5) why a three-pion $I=0$ resonance with energy well below $m_\rho+m_\pi$ should be narrow. The mass squared of the two-pion system cannot be large enough to lie in the region of the $\rho$-meson peak, and the decay occurs via the tail of the $\rho$-meson distribution. The existence of a second, lower-lying pion-pion resonance in the $J=I=1$ state would, of course, invalidate the present treatment.[8] It is interesting to note that $\kappa(s)$ is essentially equal to the decay probability corresponding to Fig. 1(a).

Finally, let us consider the effect of symmetrization on $\kappa(s)$. We used an expansion in states of the form $(12)3\rangle$, where pions 1 and 2 are combined to have angular momentum $l=1$. The following state will then have the proper symmetry: $(12)3\rangle+(23)1\rangle+(31)2\rangle$. The effect of introducing such a state is to change $\kappa(s)$ to be essentially equal to the decay matrix element calculated from the sum of the three diagrams of Fig. 1. This matrix element has been written down for both $0^-$ and $1^-$ by Shaw and Wong,[9] who pointed out that the symmetrization produces a tremendous suppression in $\kappa(s)$ for small $s$ in the $0^-$ case and electromagnetic corrections must be included if the resonance is in the neighborhood of 550 MeV. The $1^-$ matrix element calculated from any one term in Fig. 1 is already totally antisymmetric except for the $\rho$-meson propagators [the denominators in Eq. (3)]. Nevertheless the symmetrization affects the calculation of $\Gamma$. For low values of $s$ the three propagators are essentially constant and add coherently. For high values of $s$ the result is approximately the sum of the squares, so that $\kappa(s)/\bar{\kappa}(s_r)$ is raised by a factor of about 3 for small $s_r$, as compared to the unsymmetrized calculation.

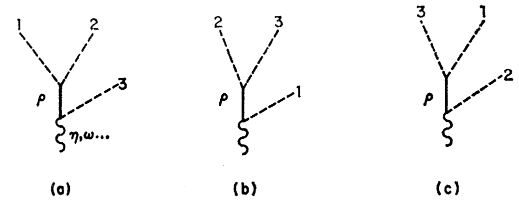


FIG. 1. Diagrams for the $3\pi$ decay of a quasi-bound-state.

The results we find for the $\omega$ ($1^-$ assignment assumed) are summarized in the table below:

| $s_0$ | Full width of $\omega$ (MeV) |
|---|---|
| 8 | 23.4 |
| 4 | 20.4 |
| 0 | 18.4 |
| −10 | 14.7 |
| −100 | 7.0 |

The disagreement between our result and that of Gell-Mann, Sharp, and Wagner[1] can probably be understood as a violation of unitary symmetry, which these authors assume in relating $\pi^0$ decay to the width of the $\omega$. Current calculations of nucleon-nucleon scattering by Scotti and Wong[10] indicate that the $\omega$-nucleon coupling is stronger by a factor of about 3 than the $\rho$-nucleon coupling; hence, the $\omega$-gamma coupling may be three times weaker than the $\rho$-gamma coupling if the electromagnetic form factors of the nucleon are dominated by the $\omega$ and $\rho$ states. This result is consistent with the calculation of the $\pi^0$ lifetime using our values of the $\omega$ width.

[8] M. Nauenberg and A. Pais, Phys. Rev. Letters **8**, 82 (1962), pointed out that the Dalitz plot of the $\omega$ decay indicates that this decay does not proceed via a low-lying pion-pion resonance.

[9] G. L. Shaw and D. Y. Wong, Phys. Rev. Letters **8**, 336 (1962).

[10] A. Scotti and D. Y. Wong (to be published).