Analyticity in Angular Momentum of the Relativistic Many-Channel S Matrix from Dispersion Relations and Unitarity*

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The analyticity of the scattering amplitude in angular momentum for N-coupled relativistic two-body channels is investigated on the basis of Mandelstam representation and unitarity. The problem of the proof of the analytic properties of the amplitude is reduced to the boundedness of a particular kernel involving the left-hand discontinuity of the amplitude. The behavior of the Regge trajectories at inelastic thresholds is determined. The results are extended to relativistic models with infinite-dimensional unitarity relation but without crossing symmetry such as the Bethe-Salpeter amplitude. The implications of the results to the exact S-matrix theory are also discussed.

I. INTRODUCTION

THE analytic properties of the scattering amplitude in the variables energy and total angular momentum has been studied extensively on the basis of wave equations.[1–4] From the point of view of the so-called S-matrix theory, the problem is to investigate the analyticity in angular momentum given the analyticity in linear momentum and unitarity. In the exact relativistic case the problem is, at present, only partially solved,[5,6] as it is difficult to make use of the full unitarity and crossing relations coupling infinitely many channels and involving infinitely many particles. In the many-channel potential scattering (and its immediate relativistic counterpart) the unitarity relation is simple but, on the other hand, because there are no crossed channels, the additional information obtained from the unitarity in crossed channels is lacking. A different method must be used than the one used in reference 5. In Sec. II the method is illustrated in terms of the single-channel problem. The left-hand discontinuity of the amplitude for any value of the complex angular momentum is given by a finite integral and is therefore a well-defined function. The unitarity gives the discontinuity across the physical cut. The problem of the analytic behavior of the amplitude in angular momentum is then reduced to the boundedness properties of a particular kernel. The case of N-coupled relativistic two-body channels is treated in Sec. III. The properties of the pole, in particular, the nature of the singularity and the threshold behavior of the Regge trajectory when it crosses an inelastic two-body threshold, are determined. Finally, in Sec. IV, the method is extended to the full unitarity condition provided the Fredholm denominator of the amplitude is invertible.

II. SINGLE-CHANNEL PROGRAM

To illustrate the method of analytic continuation in J in the absence of crossing, we consider the single-channel problem with equal masses. The potential scattering is contained as a special case. If the scattering amplitude[7] $A(s,t)$ is developed into a partial-wave expansion, one obtains in conjunction with the fixed-energy dispersion relation the following expression for the partial-wave amplitudes[5]:

$$B_t(\nu,l)=A_t(\nu,l)/\nu^l=\frac{1}{\pi}\frac{\Gamma^2(l+1)}{\Gamma(2l+2)}(4)^l\int_{4m^2}^{\infty}dt\frac{A_t(\nu,t)}{t^{l+1}}\times F(l+1,\,l+1;\,2l+2;\,-4\nu/t). \quad (1)$$

Here $A_t(\nu,t)$ is the discontinuity in t of $A(s,t)$ and F is the hypergeometric function.[8] The B amplitude is a real analytic function of ν and l, holomorphic in l for $\mathrm{Re}l>N$ under the customary hypothesis that $A_t(\nu,t)$ is bounded uniformly by t^N for large t. If exchange potential is included, we have a second amplitude $A_u(\nu,l)$ given by exactly the same expression as (1), with t replaced by u, so that the total amplitude is

$$A(\nu,l)=A_t(\nu,l)+(-1)^lA_u(\nu,l).$$

Since we have no crossing, $A_t(\nu,t)$ and $A_u(\nu,t)$ have only a right-hand cut in ν (spectral cuts). The left-hand cut of the partial-wave amplitudes $B(\nu,l)$ is therefore entirely due to that of F in Eq. (1) (projection cut).

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[1] For the single-channel Schrödinger equation, see T. Regge, Nuovo cimento **14**, 947 (1960) and A. Bottino, A. M. Longoni, and T. Regge, *ibid.* **23**, 954 (1962) in the right-hand angular momentum plane; M. Froissart (to be published); S. Mandelstam, Ann. Phys. (New York) **19**, 254 (1962); E. J. Squires, Nuovo cimento **25**, 242 (1962). H. Cheng, Phys. Rev. **127**, 647 (1962) in the whole l plane.

[2] For N-coupled two-body channel Schrödinger equation, L. Favella and M. T. Reineri, Nuovo cimento **23**, 616 (1962); A. M. Jaffe and Y. S. Kim, Phys. Rev. **127**, 2261 (1962); J. Charap and E. J. Squires, University of California Radiation Laboratory Reports UCRL-10138 and UCRL-10209 (unpublished).

[3] For Dirac equation, see L. Favella and M. T. Reineri, Nuovo cimento **23**, 616 (1962).

[4] For Bethe-Salpeter equation, see B. W. Lee and R. F. Sawyer, Phys. Rev. **127**, 2266 (1962). These authors proved the meromorphy in l plane for $\mathrm{Re}l>-3/2$.

[5] A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).

[6] K. Bardakci, Phys. Rev. **127**, 1832 (1962); G. Prosperi, Lawrence Radiation Laboratory Report, UCRL-10201 (unpublished).

[7] We use in the center-of-mass system of two equal mass particles of momentum q the variables $s=4(q^2+m^2)$, $t=-2q^2(1-\cos\theta)$, and $U=-2q^2(1+\cos\theta)$; $\nu=q^2$.

[8] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 133.

The cut of $F(a,b;c;z)$ in z plane runs from $z=1$ to $z=\infty$. Consequently, for fixed ν the discontinuity of F vanishes for $t>-4\nu$. Let us denote the left-hand discontinuity of $B(\nu,l)$ by $h(\nu,l)$, i.e.,

$$2ih(\nu,l)=B(\nu+i\epsilon,\,l)-B(\nu-i\epsilon,\,l),\quad \nu<0. \tag{2}$$

It follows from Eq. (1) that $h(\nu,l)$ is given by an integral over a finite region for all ν. Using the formula

$$\begin{aligned} Q_l(x+i0)-Q_l(x-i0)&=i\pi\bar{P}_l(x),\\ Q_l(x+i0)+Q_l(x-i0)&=2\bar{Q}_l(x), \end{aligned} \tag{3}$$

where $\bar{P}_l$ and $\bar{Q}_l$ are special solutions "on the cut" of the Legendre differential equations, defined in the interval $-1<x<+1$, and the connection between $Q_l(z)$ and F is

$$Q_l\left(1+\frac{t}{2\nu}\right)=2^l\frac{\Gamma^2(l+1)}{\Gamma(2l+2)}\left(\frac{2\nu}{t}\right)^{l+1} \times F(l+1,\,l+1;\,2l+2;\,-4\nu/t), \tag{4}$$

we first evaluate the discontinuity of F and after some calculation we obtain, with $z=1+t/2\nu$,

$$h(\nu,l)=\frac{1}{\pi}(-\nu/m)^l\int_{-1}^{1+2m^2/\nu}dz\,A_t(\nu,z) \times[\tfrac{1}{2}\pi\cos\pi l\,\bar{P}_l(z)+\sin\pi l\,\bar{Q}_l(z)];\quad \nu<0. \tag{5}$$

Clearly, $h(\nu,l)$ is an entire function in l. (The poles of $\bar{Q}_l$ in l at negative integers cancel with $\sin\pi l$.) In ν, $h(\nu,l)$ has a cut from $\nu=0$ to $\nu=-\infty$ for $l\neq$integer. The limit of $h(\nu,l)$ for $\nu=-\infty$ depends on the asymptotic behavior of $A_t(\nu,t)$ in ν. For our purpose, $h(\nu,l)$ can behave as a polynomial as $\nu\rightarrow-\infty$. In fact, for potential scattering $h(\nu,l)$ approaches zero as $\nu\rightarrow-\infty$.

We now represent the real analytic B amplitude in the well-known form of a quotient

$$B(\nu,l)=N(\nu,l)/D(\nu,l), \tag{6}$$

where N has the left cut and D the right cut of B only. We have therefore

$$N(\nu,l)=\frac{1}{\pi}\int_{-\infty}^{-m^2}d\nu'\,\frac{h(\nu',l)D(\nu',l)}{\nu-\nu'}. \tag{7}$$

There can be a number of subtractions in Eq. (7). This is immaterial for our purpose.

For $\nu>0$, the unitarity condition gives the right-hand discontinuity of B^{-1} to be

$$2ig(\nu,l)\equiv B^{-1}(\nu+i\epsilon)-B^{-1}(\nu-i\epsilon)=-\nu^l. \tag{8}$$

Hence

$$D(\nu,l)=1+\frac{(\nu-\nu_0)}{\pi}\int\frac{g(\nu',l)N(\nu',l)}{(\nu'-\nu)(\nu'-\nu_0)}d\nu'. \tag{9}$$

Again here, a number of subtractions may be present. Equations (7) and (9) can be combined giving

$$N(\nu,l)=\frac{1}{\pi}\int_{-\infty}^{-m^2}d\nu'\,\frac{h(\nu',l)}{\nu'-\nu}+\frac{1}{\pi^2}\int_0^\infty d\nu''\,\frac{g(\nu'',l)N(\nu'',l)}{\nu''-\nu_0} \times\int_{-\infty}^{-m^2}d\nu'\,\frac{h(\nu',l)(\nu'-\nu_0)}{(\nu'-\nu)(\nu''-\nu')}. \tag{10}$$

This is the well-known linear Fredholm-type integral equation of the second kind for the function $N(\nu,l)$ in ν with l as a parameter; symbolically

$$N(\nu,l)=n(\nu,l)+\int_0^\infty d\nu'\,K(\nu,\nu',l)N(\nu',l).$$

With suitable subtractions, if necessary, the integral equation is well defined. If the kernel of this equation

$$K(\nu,\nu',l)=\frac{1}{\pi^2}\frac{g(\nu',l)}{\nu'-\nu_0}\int_{-\infty}^{-m^2}d\xi\,\frac{h(\xi,l)(\xi-\nu_0)}{(\xi-\nu)(\nu'-\xi)} \tag{11}$$

is bounded, a unique continuous solution exists for $\mathrm{Re}l>N$ which can be written down explicitly:

$$N(\nu,l)=(I-K)^{-1}n(\nu,l). \tag{12}$$

Clearly there are cases where, even though the Mandelstam representation holds, the kernel (11) will not be bounded. Nothing can be concluded in these cases with the present method. Furthermore, the boundedness of the kernel cannot be proved without further assumptions on the absorptive parts A_t and A_u of the amplitude in Eq. (1). One could, of course, make suitable asymptotic assumptions on A_t which would result in a bounded kernel but this would mean, in effect, to assume what one wants to prove, namely, a Regge-type asymptotic behavior of the amplitude, hence of A_t and A_u, determined by the poles in angular momentum plane. Rather one should take the point of view that the absorptive parts are determined by the dynamics or self-consistency of the full S matrix. With such independent dynamical information one can further study the boundedness of the kernel (11). For our two-body problem we take, therefore, the boundedness of the kernel (11) as a first condition that the left-hand discontinuity of the amplitude has to satisfy to guarantee the analyticity in angular momentum. In the case of potential scattering there are, similarly, conditions on the potential under which the amplitude is meromorphic in the whole angular plane (see Froissart, reference 1).

Now in the integral equation (10) the functions $h(l)$ and $g(l)$ are entire functions. Because an integral equation can be transformed into a differential equation, the kernel being the resolvent operator, we can use the Poincaré theorem which states that if a differential equation depends in an entire fashion on a parameter l the solution is also an entire function of this parameter, provided the boundary condition does not depend on

this parameter. The unique continuous solution of the Fredholm equation is identical with this solution of the differential equation. Therefore, if the boundary condition turns out to be independent of l the solution $N(\nu,l)$ is an entire function of l; otherwise one can read off the analyticity from the Eq. (12) in which case $N(\nu,l)$ could have poles due to the vanishing of the denominator. Once $N(\nu,l)$ is determined, the function $D(\nu,l)$ can be evaluated by Eq. (9). The amplitude B is then, under the conditions stated, a meromorphic function in the product of the whole l plane with the cut energy plane.

It should be remarked that there are cases such as the square well potential in the potential scattering, for example, for which the amplitude is meromorphic in the whole angular momentum plane, but does not satisfy the Mandelstam representation. The analytic continuation in l can no longer be defined by Eq. (1). But we know that in this case the analytic continuation is not unique.[9] We can take $A_0(\nu,l)=\frac{1}{2}\int_{-1}^{+1}dzA(\nu,z)P_l(z)$, which has no poles in l, or a whole class of functions

$$A_0(\nu,l)+\sin\pi l f(\nu,l),$$

where $f(\nu,l)$ is so chosen that the Watson-Sommerfeld transformation is possible within a cone in the right-hand l plane.

III. COUPLED RELATIVISTIC TWO-BODY CHANNELS

We now consider N two-body channels of spinless particles.[10] The scattering amplitude is an N-by-N matrix, A_{ij}. In each channel $i\rightarrow j$ with masses $m_{1,i}$, $m_{2,i}$, and $m_{3,j}$, $m_{4,j}$ the equation corresponding to Eq. (1) is

$$A_{ij}(s,J)=\frac{1}{\pi}\frac{\Gamma^2(J+1)}{\Gamma(2J+2)}(4q_iq_j)^J\int_{t_0}^{\infty}dt\frac{A_t^{(ij)}(s,t)}{[t+2q_iq_j(a_{ij}-1)]^{J+1}}$$
$$\times F\left(J+1,\,J+1,\,2J+2;\,-\frac{4q_iq_j}{t+2q_iq_j(a_{ij}-1)}\right),\quad(13)$$

where q_i, q_j are the momenta in the center-of-mass system of the incoming and outgoing particles and

$$a_{ij}=[2(q_i^2+m_{1,i}^2)^{1/2}(q_j^2+m_{3,j}^2)^{1/2}-m_{1,i}^2-m_{3,j}^2]/2q_iq_j,$$

which is 1 for elastic scattering. The momenta q_i and q_j must be expressed in terms of the energy squared s on the left-hand side by

$$s=m_1^2+m_2^2+2q_i^2+2(q_i^2+m_1^2)^{1/2}(q_i^2+m_2^2)^{1/2},$$
$$=m_3^2+m_4^2+2q_j^2+2(q_j^2+m_3^2)^{1/2}(q_j^2+m_4^2)^{1/2}.$$

We see from Eq. (11) that the matrix

$$B_{ij}(s,J)=A_{ij}(s,J)/(q_iq_j)^J\quad(14)$$

is a real analytic function of s and J. A mass factor $(m_{1j}m_{3j})^{J/2}$ may be introduced on the right for dimensional reasons. Again, the left-hand discontinuity $h_{ij}(s,J)$ of $B_{ij}(s,J)$ in s is due to the discontinuity of F, which, in turn, vanishes for $t>-4q_iq_j-2q_iq_j(a_{ij}-1)$, $s<0$. Therefore, $h_{ij}(s,J)$ is given by a finite integral of the type of Eq. (5) and is again an analytic function of s and holomorphic function of J. The only difference is now in the unitarity relation coupling all the channels. Instead of the Eq. (6), we now use the matrix ND^{-1} method.[11] We define a matrix D by

$$D=1+\frac{1}{\pi}\int\frac{\bar{\rho}N}{s'-s}ds',\quad(15)$$

where the integral is over any finite part or the whole of the physical cut. The matrix $\bar{\rho}N$, with the diagonal matrix $\bar{\rho}$ given below, has by definition no cut for physical s values. It follows from (15) that the discontinuity of D across the right cut is equal to $\bar{\rho}N$. It will now be shown that the amplitude B defined by

$$N(s,J)=B(s,J)D(s,J)\quad(16)$$

satisfies the unitarity condition. If we denote by s_+ and s_- the value of s just above and below the right-hand cut, we have (omitting the J index)

$$0=N(s_+)-N(s_-)=B(s_+)D(s_+)-B(s_-)D(s_-)$$
$$=B(s_+)[D(s_-)+2i\bar{\rho}]-B(s_-)D(s_-),$$

or

$$[B(s_+)-B(s_-)]D(s_-)=-2iB(s_+)\bar{\rho}B(s_-)D(s_-).\quad(17)$$

Since D is a finite-dimensional matrix, the inverse exists except at a finite number of isolated points. Furthermore, since B is a real analytic function, we have

$$B(s_-)=B^\dagger(s_+).\quad(18)$$

Consequently, we obtain

$$B(s_+)-B^\dagger(s_+)=-2iB(s_+)\bar{\rho}B^\dagger(s_+),\quad(19)$$

which is the unitarity condition for the B amplitude. From the unitarity condition for A amplitudes and Eq. (14), we obtain that

$$\bar{\rho}_{ii}=-\frac{2q_i^{2J+1}}{(q_i^2+m_{1,i}^2)^{1/2}+(q_i^2+m_{2,i}^2)^{1/2}}.\quad(20)$$

In Eq. (19), $\bar{\rho}$ is also understood to be multiplied by $\theta(S-M_i)$, where $M_i=m_{1,i}+m_{2,i}$ is the total channel mass.

We have thus constructed amplitudes satisfying the unitarity condition by Eq. (16). Since the elements of N are given in terms of their left-hand cut discontinuities by integral equations of the form (10), the elements of

[9] A. O. Barut and F. Calogero, Phys. Rev. **128**, 1383 (1962).

[10] For the analytic continuation in J of two-body reactions with arbitrary spins, see A. O. Barut, I. Muzinich, and D. Williams, University of California Radiation Laboratory Report UCRL-10463 (unpublished).

[11] N. D. Bjorken, Phys. Rev. Letters **4**, 473 (1960); J. D. Bjorken and M. Nauenberg, Phys. Rev. **121**, 1250 (1961); R. Blankenbecler, *ibid.* **122**, 983 (1961).

D can then be determined by Eq. (15). This proves the meromorphy of $B(s,J)$ in the whole J plane, again to the extent that the kernel of (10) is bounded.

In order to get more information on the properties of the poles, we rewrite the inverse scattering amplitude in terms of the so-called Y matrix[5] which is the analytic continuation in J of the inverse reaction matrix K^{-1}. For this purpose, we first obtain from Eq. (19) the discontinuity of the B^{-1} amplitude across the physical cut:

$$B^{-1}(s_+,J)-B^{-1}(s_-,J)=-2i\bar{\rho}(s,J). \qquad (21)$$

We can therefore write

$$(B^{-1})_{ij}(s,J)=Y_{ij}(s,J)+\frac{2(\cos\pi J)^{-1}}{(q_i^2+m_1^2)^{1/2}+(q_i^2+m_2^2)^{1/2}} \times (q_i^2)^{J+1/2}e^{-\pi(J+1/2)}\delta_{ij}$$

or in matrix form

$$B^{-1}=Y+R, \qquad (22)$$

where the second diagonal matrix is so constructed that it has the given discontinuity (17) across the physical cut and the matrix Y has only the left-hand cut which, in the absence of crossing, is just the projection cut. Inverting Eq. (22) w get

$$B=\frac{\mathrm{adj}(Y+R)}{\det(Y+R)}. \qquad (23)$$

The poles of the amplitude are associated with the zeros of the determinant of $Y+R$. Therefore the same pole will occur in every channel. Furthermore, the residue of the pole—if it is a simple pole—factorizes, i.e.,

$$\mathrm{adj}(Y+R)_{ij}|_{\mathrm{pole}}=g_ig_j. \qquad (24)$$

These results have been already noticed in the case of nonrelativistic coupled channel problem.[12]

Since Y has no singularity along the physical region, the equation $\det(Y+R)=0$ can be used to determine the threshold behavior of the poles by expanding $\det(Y+R)$ near $q_i=0$. Suppose we are looking at the ith threshold. Since the determinant is linear in each element, we have

$$\det(Y+R)=\det(Y')+R_{ii}[Y+R]_{ii}=0,$$

where Y' is the matrix obtained from $Y+R$ by deleting a single term R_{ii}, and $[Y+R]_{ii}$ is the cofactor of the element ii. With Eq. (22) we can therefore write the pole equation as

$$y^{(i)}(q_i^2,J)+(q_i^2)^{J+1/2}e^{-i\pi(J+1/2)}=0, \qquad (25)$$

where the scalar function

$$y^{(i)}=\tfrac{1}{2}\cos\pi J[(q_i^2+m_1^2)^{1/2}+(q_i^2+m_1^2)^{1/2}]\frac{\det(Y')}{[Y+R]_{ii}}$$

has no branch point at $q_i=0$ and can therefore be expanded in q_i^2 and J. Let $J_0^{(i)}$ be the position of the pole at $q_i^2=0$ and $\mathrm{Re}J_0>-\frac{1}{2}$. Then $y^{(i)}(0,J_0)=0$ and we get, with $\nu_i=q_i^2$,

$$J=J_0^{(i)}+a^{(i)}\nu_i+O(\nu_i^2)+b^{(i)}\nu_i^{J+1/2}e^{-i\pi(J+1/2)}. \qquad (26)$$

Thus, the threshold behavior of a Regge trajectory at the beginning of each inelastic two-body threshold is essentially the same as at the elastic threshold.[5] The only difference is that the constants $J_0^{(i)}$, $a^{(i)}$, and $b^{(i)}$ are complex except at the elastic threshold. For $J_0>+\frac{1}{2}$ the slope of the trajectory is continuous at these inelastic thresholds.

The full S-matrix theory differs in two important respects from the case we have considered in this section. Firstly, we have to allow in the unitarity relation intermediate states with arbitrarily many particles. Secondly, the partial-wave amplitudes will have, in addition to the projection left cuts, also left-hand cuts due to the spectral functions of the crossed channels. In the following section, we remove the first restriction under certain conditions. With respect to the second point it should be noted that even the spectral left-hand cut of $B(\nu,l)$ in Eq. (1) is given by a finite integral. It is proportional to

$$\int dt\, Q_l\left(1+\frac{t}{2\nu}\right)\rho_{tu}(t,\,-4\nu-t),$$

where for fixed and negative ν the spectral function ρ_{tu} is different from zero over a finite region in t. This discontinuity, however, has the poles of Q_l at negative integers and does not produce a bounded kernel at these points.

IV. FULL UNITARITY

We now discuss under what conditions the proof of the previous section may be extended to the full unitarity condition involving intermediate states with arbitrarily many particles. It is plausible that there will be a real analytic B amplitude constructed by a generalization of Eq. (14). The form of the unitarity is still given by Eq. (19) where B is now a continuously infinite dimensional matrix, $B(E,J)$; the remaining matrix indices being suppressed. These are various angle, spin, and relative energy variables. We can define the D matrix again by Eq. (15). Then Eqs. (19) and (21) hold, provided D has an inverse. The conditions for the inverse of continuously infinite matrices is given by Fredholm theory. For the inverse of D to exist, it is sufficient that the kernel

$$\frac{1}{\pi}\int\frac{\bar{\rho}N}{E'-E}dE'$$

of the Fredholm equation of the second kind is bounded. This is the case if the amplitude consists solely of the

[12] M. Gell-Mann, Phys. Rev. Letters, **8**, 263 (1962). J. Charap and E. Squires, Phys. Rev. **127**, 1387 (1962). For a general proof of factorizability of poles residues in the second sheet of energy in relativistic case, see reference 13.

connected diagrams in which all particles interact.[13] If in some S-matrix element a subclass of particles do not interact with the rest of the particles, there will be additional momentum conservation δ functions for the subclass in N, so that it is not clear whether the inverse of D still exists.

The elements of N have left-hand cuts in energy which are due to the partial-wave projections only, other variables being fixed. Their discontinuity will again be given by a finite integral as before. One can therefore safely conclude that for relativistic model theories which sum connected diagrams only and have no crossing symmetry, the amplitude is meromorphic in the whole J plane. Singularities in l plane can arise only in connection with the disconnected S-matrix elements or through the spectral left-hand cuts due to crossed channels. In order to treat the crossing exactly, one has to write the unitarity in cross channels, say, again in terms of energy E' and total angular momentum J'. The relation of the variables in the crossed channels to that of the original channel, in the general case, is the important problem of the S-matrix theory yet to be solved.

[13] H. P. Stapp, "On the Masses and Lifetimes of Unstable Particles" (to be published).

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Intrinsic Parity from the S-Matrix Viewpoint

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The concept of intrinsic parity is developed as a construct of observables within the S-matrix framework, and ambiguities arising in more conventional approaches are avoided. It is shown that invariance of transition probabilities under spatial inversion implies the existence of a set of particle intrinsic parities that is real and unique to within a gauge transformation, provided all processes not forbidden by additive conservation laws do occur in nature. If certain of the conservation laws are multiplicative, then nonreal intrinsic parities are possible. The intrinsic parity of particle-antiparticle pairs is fixed by the general S-matrix postulates to be negative for fermions and positive for bosons, a result that parallels the one of field theory.

I. INTRODUCTION

THE question whether the sign of the intrinsic parity of particle-antiparticle pairs follows directly from the S-matrix postulates has served to focus attention on the more general question of the meaning of intrinsic parity in the S-matrix framework. In this paper the concept of intrinsic parity is systematically developed from observable S-matrix quantities. The approach is more general than the usual one, as the existence of a parity operation in the abstract space of field operators is not assumed; the only assumed invariance is that of the physically observed transition probabilities. The notion of an intrinsic parity for individual particles is not a manifest part of this assumption, but the concept can be established by construction, and the questions of existence and uniqueness completely answered. The problem of superselection rules, which clouds the conventional approach, causes no difficulty in this one. A more detailed comparison with the conventional approach is given in the final section.

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II. INVARIANCE OF A PROCESS UNDER SPATIAL INVERSION

A *process* is taken to mean a reaction having specified initial and final particles. A process will be said to be *invariant* under spatial inversion (reflection) if and only if the transition probabilities associated with the process are invariant under an inversion through the origin of all polar three-vectors. The space parts of momentum-energy vectors are defined to be polar three-vectors, whereas spin vectors are not polar vectors. Thus, in the S-matrix framework, invariance under spatial reflection implies the equation[1,2]

$$|R(K_p)|=|R(K)|, \tag{2.1}$$

where K_p is the set of variables obtained from the set K by reversing the space parts of all the momentum-energy vectors k_i contained in K. This equation can also

[1] Henry P. Stapp, Phys. Rev. **125**, 2139 (1962) and *Lectures on S-Matrix Theory* (W. A. Benjamin Inc., New York, 1963). These references will be referred to as SI and SII, respectively.

[2] See SI, Appendix E.