PHYSICAL REVIEW VOLUME 128, NUMBER 5 DECEMBER 1, 1962

Stability of Force-Free Magnetic Fields*

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(Received March 23, 1962; revised manuscript received August 20, 1962)

A formalism for treating the stability of force-free magnetic fields ($\nabla\times\mathbf{B}=\alpha\mathbf{B}$) based on the energy principle of Bernstein *et al.* is derived. An example is given to shown that force-free fields with constant α may be unstable, though previous papers give arguments in favor of the stability of such fields. It is found that the cylindrically symmetric force-free field given by Lundquist is unstable if and only if αr_0 (where r_0 is the radius of the cylinder) exceeds the critical value 3.176. The unstable displacements have small growth rates. They are of the screw or kink type, their wavelengths along the axis having a minimum of about seven times the critical cylinder radius.

The results are applied to the arms of spiral galaxies, correcting Trehan's statement that some force-free fields with constant α have a stabilizing effect on gravitational instability.

I. INTRODUCTION

An important special class of hydromagnetic equilibria are those characterized by the vanishing of the Lorentz force. The magnetic fields of such configurations obey the equation

$$\mathrm{curl}\mathbf{B}=\alpha\mathbf{B}, \tag{1a}$$

where α is a scalar function of space satisfying

$$\mathbf{B}\cdot\mathrm{grad}\alpha=0. \tag{1b}$$

Because of the great interest of force-free fields in astrophysics and, more recently, also in experimental plasma physics, their hydromagnetic stability has been examined by many authors. It has generally been found that the case $\alpha=\mathrm{const}$ plays a particular role, as it seemed to be most favorable for stability. Until now no case of instability had been discovered for constant α except those due to self-gravitational forces. In most of these investigations, more or less general configurations with constant α are proved to be stable against more or less wide classes of perturbations.

Lundquist[1] showed that the cylindrically symmetric field

$$B_r=0,\quad B_\varphi=B_0J_1(\alpha r),\quad B_z=B_0J_0(\alpha r) \tag{2}$$

(r, φ, z cylindrical coordinates, J_0, J_1 Bessel functions) is stable against perturbations of the form

$$\xi=(c\cos bx\sin az,\quad 0,\quad c'\sin bx\cos az)$$

(x, y, z Cartesian coordinates).

Trehan[2] states that the gravitational instability of the arms of spiral galaxies is decreased by the presence of a certain type of force-free magnetic fields which contain Lundquist's field (2) as a special case. However, his proof is not complete since he considers a very restricted class of perturbations, namely, axisymmetric and irrotational ones.

The stability of general force-free fields with constant α against spherically symmetric expansions has been shown by Woltjer.[3] He further proves that equilibria with axial symmetry are stable against all axisymmetric perturbations, the normal component of which vanishes on the surface of the field-containing region.

In another paper,[4] Woltjer states that for the case of constant α the first-order variation of the magnetic energy with an appropriate constraint vanishes for closed systems. He concludes in a later article[5] that this means stability for the lowest possible value of α compatible with the geometry considered. The other possible values of α correspond to extrema of an unknown nature. Only by determining the second-order variation of the energy can it be established whether these extrema are relative minima and thus stable to small perturbations.

In view of all this evidence, one might be tempted to conclude even in the absence of a rigorous proof that force-free fields with constant α are stable. However, this is not true at all and, in fact, it is the purpose of this paper to show that there is a rather wide class of force-free fields with constant α which are indeed unstable.

II. THE SECOND-ORDER ENERGY VARIATION FOR FORCE-FREE FIELDS

We consider a plasma in static equilibrium enclosed by a perfectly conducting rigid wall, on which the normal component of the magnetic field vanishes. The plasma has infinite conductivity and is treated in the hydromagnetic approximation.

According to the energy principle of Bernstein *et al.*,[6] the system is stable if and only if the second-order energy variation

$$\delta W=\frac{1}{2}\int[\mathbf{Q}^2-\mathbf{j}\cdot\mathbf{Q}\times\xi+\gamma p(\mathrm{div}\xi)^2+(\mathrm{div}\xi)(\xi\cdot\mathrm{grad}p)]dv \tag{3}$$

is positive definite to perturbations ξ whose normal

* This work was supported by Euratom.

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[1] S. Lundquist, Phys. Rev. **83**, 307 (1951).
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[5] L. Woltjer, Proc. Natl. Acad. Sci. U. S. **45**, 769 (1959).
[6] I. B. Bernstein, E. A. Frieman, M. D. Kruskal, and R. M. Kulsrud, Proc. Roy. Soc. (London) **A244**, 17 (1958).

component vanishes on the surface. [$\mathbf{Q}=\mathrm{curl}(\xi\times B)$, $\mathbf{j}$ denotes the current density, p the pressure, and γ the ratio of the specific heats. The integration is extended over the plasma volume.]

For force-free fields given by Eq. (1), where α need not be a constant, this expression becomes

$$\delta W=\delta W_1+\delta W_2, \tag{4}$$

where

$$\delta W_1=\frac{1}{2}\int[(\mathrm{curl}\mathbf{R})^2-\alpha\mathbf{R}\cdot\mathrm{curl}\mathbf{R}]dv, \tag{5}$$

$$\delta W_2=\gamma p\int(\mathrm{div}\boldsymbol{\xi})^2dv, \tag{6}$$

$$\mathbf{R}=\boldsymbol{\xi}\times\mathbf{B}. \tag{7}$$

In order to find the most unstable displacements ξ, δW should be minimized under a normalization condition which makes δW bounded from below, thus guaranteeing that a minimum exists.

The Euler equations resulting from the energy principle usually form a system of sixth order. In our case we can reduce it to fourth order by limiting ourselves to minimizing only the magnetic part and by choosing a variational constraint which contains only $\mathbf{R}$ and not ξ explicitly. This involves no essential loss in generality, for if the minimum of δW_1 is positive, the system is stable since the additional term δW_2 is always positive. If the minimum of δW_1 is negative, we must take into account the stabilizing term to determine whether the system is really unstable. In many cases, this term may vanish already since $\delta W_{1\min}$ is attained for a divergenceless displacement ξ. In the other cases there is some critical value p_c of the constant pressure below which there is instability. Thus each magnetic configuration for which $\delta W_{1\min}$ is negative is unstable when the pressure is sufficiently small.

Let us minimize δW_1 [Eq. (5)] with the constraint

$$-\int\alpha\mathbf{R}\cdot\mathrm{curl}\mathbf{R}dv=\mathrm{const}\neq0, \tag{8}$$

for which δW_1 is obviously bounded from below. We thus obtain the Euler equations,

$$\mathbf{B}\times[\mathrm{curl}\,\mathrm{curl}\mathbf{R}-(\lambda+1)\alpha\,\mathrm{curl}\mathbf{R}]=0, \tag{9}$$

where the constant Lagrangian multiplier λ corresponds to (8). It has to be determined as an eigenvalue of Eq. (9) under the condition

$$\mathbf{R}\cdot\mathbf{B}=0, \tag{10}$$

which comes from (7) and the boundary condition

$$\mathbf{n}\times\mathbf{R}=0 \tag{11}$$

($\mathbf{n}$ is the unit normal vector to the surface). Equation (11) means that on the surface the displacement has to be tangential to the rigid wall.

The condition (8) is not a normalization in the usual sense as it excludes all $\mathbf{R}$ which give the integral in (8) another sign than that of the constant on the right-hand side of this equation. However, the value of the constant does not enter into the Euler equations (9) which therefore are generally valid, regardless of the sign of the normalization constant. We may consider this constant to be -1, as we are interested in finding possible cases of instability.

Note that in our variational calculation we may replace condition (8) by the normalization

$$\int[\mathrm{curl}\mathbf{R}]^2dv=N, \tag{12}$$

since this yields Euler equations which differ from (9) by a simple transformation of the eigenvalues. Denote by μ the Lagrangian multiplier corresponding to (12). Then we have the relation

$$(\lambda+1)(\mu+1)=1, \tag{13}$$

which states a one-to-one correspondence between the eigenvalues λ and μ belonging to the same eigenfunction ($\mu=-1$ and $\lambda=-1$ should be excepted; however, for these cases $\delta W_{1\min}$ does not become negative).

Let $\mathbf{R}_0$ be a solution of (9) satisfying conditions (8), (10), and (11). Then we find with the help of (13) that

$$\delta W_1=-\mu\int[\mathrm{curl}R_0]^2dv=-\mu N. \tag{14}$$

The system therefore is unstable if and only if the largest eigenvalue μ is positive.

By introducing two vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ perpendicular to $\mathbf{B}$ and by defining

$$\mathbf{R}=R_1\mathbf{n}_1+R_2\mathbf{n}_2, \tag{15}$$

we can write Eqs. (9) in the following manner:

$$\begin{aligned}&\mathbf{n}_1\cdot\{\mathrm{curl}\,\mathrm{curl}(R_1\mathbf{n}_1+R_2\mathbf{n}_2)\\&\qquad-[\alpha/(\mu+1)]\,\mathrm{curl}(R_1\mathbf{n}_1+R_2\mathbf{n}_2)\}=0,\\&\mathbf{n}_2\cdot\{\mathrm{curl}\,\mathrm{curl}(R_1\mathbf{n}_1+R_2\mathbf{n}_2)\\&\qquad-[\alpha/(\mu+1)]\,\mathrm{curl}(R_1\mathbf{n}_1+R_2\mathbf{n}_2)\}=0.\end{aligned} \tag{16}$$

These equations are independent of the component $R_{||}$ of $\mathbf{R}$ parallel to the magnetic field. $R_{||}$ may be chosen to make $|\mathrm{div}\xi|$ as small as possible. However, it is possible to make $\mathrm{div}\xi\equiv0$ if and only if the "magnetic differential equation" (Newcomb[7])

$$\mathbf{B}\cdot\mathrm{grad}\eta=\mathrm{div}\left(\frac{\mathbf{R}\times\mathbf{B}}{|\mathbf{B}|^2}\right)$$

has a solution η in the interior of the plasma volume.

[7] W. A. Newcomb, Phys. Fluids 2, 362 (1959).

The condition that this equation have a solution is

$$\oint \frac{1}{|\mathbf{B}|}\,\mathrm{div}\left(\frac{\mathbf{B}\times\mathbf{R}}{|\mathbf{B}|^2}\right)dl=0,$$

where the integration is over each closed line of force.

Equations (16) are still valid if α is not constant and may be a starting point for the general treatment of force-free stability problems. In the next section we shall solve them exactly for the special case of a cylindrically symmetric magnetic field and constant α.

III. SPECIAL CASE OF CYLINDRICAL GEOMETRY

We take the model of an infinitely long plasma cylinder. This may be considered as the limiting case of a very thin torus or a very long plasma column with electrodes at the ends. By assuming the magnetic field to be axisymmetric and to have no radial component, we obtain Lundquist's solution (2) of Eqs. (1) for constant α. The radius r_0 of the column wall being fixed, α can still take continuously all values from zero to infinity, contrary, for example, to the case of spherical equilibria.

In applying Eqs. (16) to the magnetic configuration (2), choose

$$\mathbf{n}_1=-J_0(\alpha r)\mathbf{n}_\varphi+J_1(\alpha r)\mathbf{n}_z,\quad \mathbf{n}_2=\mathbf{n}_r, \tag{17}$$

where $\mathbf{n}_r$, $\mathbf{n}_\varphi$, $\mathbf{n}_z$, denote the cylindrical unit vectors. By the Fourier representation

$$R_1=\sum_{k,m} s_{km}(r)e^{i(kz+m\varphi)},\quad R_2=\sum_{k,m} q_{km}(r)e^{i(kz+m\varphi)}, \tag{18}$$

equations (16) lead to an infinite set of independent ordinary differential equations:

$$\frac{d}{dr}\left[f_{km}\frac{ds_{km}}{dr}\right]-g_{km}s_{km}=0, \tag{19}$$

where

$$f_{km}=\frac{r(krJ_0+mJ_1)^2}{k^2r^2+m^2}, \tag{20a}$$

$$\begin{aligned} g_{km}=&\frac{1}{r}\frac{(k^2+\alpha^2-\beta^2)r^2+m^2-1}{k^2r^2+m^2}(krJ_0+mJ_1)^2\\ &+\frac{2k^2r}{(k^2r^2+m^2)^2}(k^2r^2J_0^2-m^2J_1^2)\\ &+\frac{2k(\alpha-\beta)mr}{(k^2r^2+m^2)^2}(krJ_0+mJ_1)^2, \end{aligned} \tag{20b}$$

$$\beta=\alpha/(\mu+1). \tag{21}$$

The connection between s_{km} and q_{km} is given by

$$\begin{aligned} iq_{km}=&\frac{1}{k^2r^2+m^2}\{r(krJ_1-mJ_0)s_{km}'\\ &+[(\alpha-\beta)r(krJ_0+mJ_1)-(krJ_1+mJ_0)]s_{km}\}. \end{aligned} \tag{22}$$

It is interesting to compare the expressions for f_{km} and g_{km} with those in the equations which Newcomb[8] has derived for the general case of stability of cylindrically symmetric hydromagnetic configurations. If in our equations we take $\mu=0$ then β equals α, and they agree exactly with Newcomb's equations for our special case. This is to be expected since Newcomb does not apply a normalization.

With the transformation

$$s_{km}=y_{km}/(krJ_0+mJ_1), \tag{23}$$

Eq. (19) becomes

$$\begin{aligned} y_{km}''+y_{km}'\frac{1}{r}\left(1-\frac{2k^2r^2}{k^2r^2+m^2}\right)\\ +y_{km}\left(\beta^2-\alpha^2-\frac{m^2}{r^2}+\frac{2m\beta k}{k^2r^2+m^2}\right). \end{aligned} \tag{24}$$

The general solution of this equation is

$$\begin{aligned} y_{km}=\frac{k}{\beta-k}r(\beta^2-k^2)^{1/2}Z_{m-1}[r(\beta^2-k^2)^{1/2}]\\ +mZ_m[r(\beta^2-k^2)^{1/2}]. \end{aligned} \tag{25}$$

Here Z_m and Z_{m-1} are linear combinations of the Bessel and Neumann functions:

$$\begin{aligned} Z_{m-1}&=c_{1km}J_{m-1}+c_{2km}N_{m-1},\\ Z_m&=c_{1km}J_m+c_{2km}N_m. \end{aligned} \tag{26}$$

With the help of (23), (25), and (22), the Fourier series (18) can now explicitly be written down.

We see from (14) and (21) that instability occurs if and only if there are solutions $R_1(r,\varphi,z)$ and $R_2(r,\varphi,z)$ which belong to values of β with $0<\beta<\alpha$. If R_1 is such a function, it is clear that each of its nonvanishing Fourier harmonics must be so, too. Thus it is possible to determine stability directly from (23) and (25). Due to condition (11) s_{km} must vanish at $r=r_0$ (the wall).

However, the situation is not completely straightforward. We note that from (23) s_{km} will have a singularity wherever the denominator vanishes. The corresponding values of r define "singular surfaces" which divide the cylinder into a finite number of concentric zones. The integrations in (5) and (8) would diverge if they were extended over the singular surfaces. In order to apply the results of our variational calculation separately to the different regions between the singularities, we make use of the following theorem:

For fixed values of k and m the whole system is stable if and only if (a) each region between two singular surfaces and between the last singular surface and the wall is stable to perturbations whose radial component vanishes in its interior for at least two values of r, and

[8] W. A. Newcomb, Ann. Phys. (New York) **10**, 232 (1960).

(b) the region within the first singular surface is stable to perturbations which vanish in its interior for at least one value of r.

The "only if" part of this theorem is almost trivial. The "if" part can be proved by calculating δW_1 from formula (5) for the single Fourier components in (18). The resulting integration may be carried out with respect to z and φ, thus yielding a quadratic form in s_{km}, s_{km}', and q_{km}. Suppose now that δW_1 be negative for some s_{km} and q_{km}. Splitting up the integral according to the different regions between the singularities, one has clearly at least one region which gives the integral a negative contribution. In general, the corresponding perturbation (a part of the initial perturbation defined in the whole cylinder) does not satisfy the conditions formulated in (a) or (b) of the theorem. But we can use this perturbation to construct a new one by two steps: First keep s_{km} the same and replace q_{km} by the expression (22) with $\beta=\alpha$. As this means a minimization of δW_1, the integral, which is now a quadratic form in s_{km} and s_{km}' only, will still be negative. The coefficient of $s_{km}'^2$ is f_{km} as given in formula (20a) and it has second-order zeros at the singular surfaces. The latter fact enables us to carry out the second step, i.e., to change s_{km} in an arbitrarily small vicinity of the singular point in such a way that (1) s_{km} vanishes inside the region and (2) δW_1 is changed by a sufficiently small amount that it remain negative. (s_{km} remains the same everywhere else in the region.) Thus δW_1 would still be negative for perturbations as described in (a) and (b), Q. E. D.

In view of this theorem, we have to examine, for each couple (k,m), the stability of all the intervals which are defined by the singular points of the Euler solution (23). In each case we must find out whether there are values of $\beta<\alpha$ for which the conditions stated in (a) and (b) can be satisfied by a function equal to the Euler solution with an appropriate choice of c_{1km} and c_{2km} between the two values of r and zero elsewhere. If $\beta<\alpha$ is possible the situation is unstable, otherwise stable.

Let us first examine whether the interval between the axis and the first singularity may be unstable, if k and m is properly chosen. We see from regularity at the origin that c_{2km} must vanish. Thus we obtain from (23), (25), and (26)

$$s_{km}=\frac{[k/(\beta-k)]r(\beta^2-k^2)^{1/2}J_{m-1}[r(\beta^2-k^2)^{1/2}]+mJ_m[r(\beta^2-k^2)^{1/2}]}{krJ_0(\alpha r)+mJ_1(\alpha r)}. \tag{27}$$

Further, we have the instability condition that s_{km} vanish once in the interval considered, i.e., that its first zero lies before its first singularity. To take this into account, we introduce the dimensionless quantities

$$x=\alpha r,\quad \kappa=k/\alpha,\quad b=\beta/\alpha, \tag{28}$$

and examine separately for each integer m, for which regions of κ the smallest value x_N defined by

$$\kappa x_N J_0(x_N)+mJ_1(x_N)=0 \tag{29}$$

is larger than the smallest value x_Z defined by

$$\frac{\kappa}{b-\kappa}x_Z(b^2-\kappa^2)^{1/2}J_{m-1}[x_Z(b^2-\kappa^2)^{1/2}] +mJ_m[x_Z(b^2-\kappa^2)^{1/2}]=0. \tag{30}$$

The definition of x_Z still contains the parameter b which for the unstable cases must take values between 0 and 1. For the regions of κ where x_Z may be less than x_N, the smallest x_Z occurs for $b=1$ (marginal stability). It follows from the properties of the Bessel functions with pure imaginary argument that expression (27) has no zeros for $\kappa^2>b^2$, except $r=0$. If therefore there are any cases of instability they can only occur for $\kappa^2<1$.

In Fig. 1 the functions $\kappa=\kappa(x_N)$ and $\kappa=\kappa(x_Z)$ as defined by (29) and (30) have been drawn for the cases $m=1$ and $m=2$. The diagram shows that only for $m=1$ is there a small region for which $x_Z<x_N$ holds, thus indicating instability. The dashed curve $\kappa(x_Z)$ cuts the curve $\kappa(x_N)$ at the two points $\kappa=0.272$, $x=3.176$ and $\kappa=-0.237$, $x=4.744$. Between these points $\kappa(x)$ is on the left of $\kappa(x_N)$ except at the point $\kappa=0$, $x=3.832$, where the two curves are tangent to each other.

The difference x_Z-x_N is very small in the region where it is negative, and great precision is needed to determine the region of instability. However, one can see by a series expansion that for small κ, x_Z is indeed less than x_N: Equations (29) and (30) can be explicitly solved for x_N and x_Z in the neighborhood of $\kappa=0$. The first nonvanishing term of the expansion of the differ-

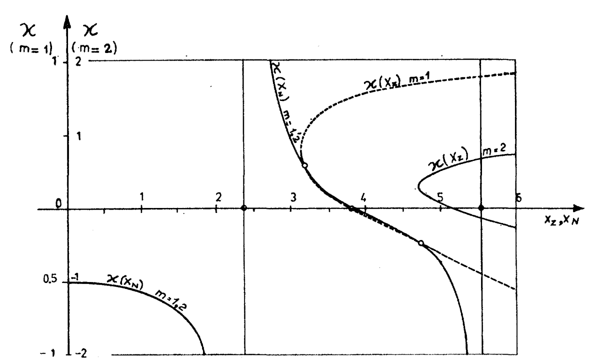

FIG. 1. A plot of the first zero x_Z and the first singularity x_N of expression (27) for $m=1$ and $m=2$ as functions of κ. The solid lines represent the singularities and the dashed lines the zeros. (Note that the ordinate is κ and the abscissa is x_N or x_Z.) κ has a different scale for $m=1$ and $m=2$. The unstable range of κ is the range for which the dashed line lies to the left of the solid line. This occurs for $m=1$.

ence is quadratic and negative:

$$x_Z - x_N \approx -\tfrac{1}{2} j_{11} \kappa^2, \tag{31}$$

where j_{11} is the first zero of $J_1(x)$.

For the case $m=0$, which is not represented in Fig. 1, one gets stability. This is almost trivial, as for $b^2 \leq 1$ the first zero of $J_0(x)$ is always less that the first zero of $J_1[x(b^2-\kappa^2)^{1/2}]$. For the cases $m \geq 2$ one gets stability, too, since the curves $\kappa(x_Z)$ in Fig. 1 are more and more displaced to the right as m grows and therefore have no more intersections with the corresponding curves $\kappa(x_N)$. For the first interval we thus find instability only for $m = \pm 1$, $|\kappa| \leq 0.272$,[9] and only if the radius of the plasma column is so large that the first zero x_Z of (30) can lie in the interior of the plasma or at the wall.

The least value r_{0c} of the radius for which the instability can occur is given by

$$\alpha r_{0c} = 3.176. \tag{32}$$

At the point r_{0c} the azimuthal component of equilibrium current and of magnetic field still have the same direction as in the neighborhood of the axis, while the longitudinal component has reversed its direction. Since

$$\alpha^2 = \left(\frac{1}{2}\int \mathbf{j}^2 dv\right) \Big/ \left(\frac{1}{2}\int \mathbf{B}^2 dv\right), \tag{33}$$

where the numerator is proportional to the mean square current density, and the denominator is the magnetic energy, Eq. (32) defines a critical value for the mean square current density above which the plasma is unstable.

So far we have investigated only the interval between the axis and the first singular point. Now there is the question whether by extending our treatment to the interval between the first and the second singularity we might obtain a smaller critical value αr_{0c} instead of (32). This would be the case if for some k and m the constants c_{1km} and c_{2km} could be chosen in such a way that s_{km} has two zeros between its first and its second singularity, where the larger zero must still be less than the critical radius r_{0c} as defined by (32). However, this is never possible as can be shown for $\kappa^2 \leq 1$ with the help of Fig. 1. The proof can be given first for the special case $c_{2km}=0$ and next by taking into account that two solutions y_{km} of (24), one with $c_{2km}=0$ and the other with $c_{2km} \neq 0$, are linearly independent and thus have mutually separated zeros.

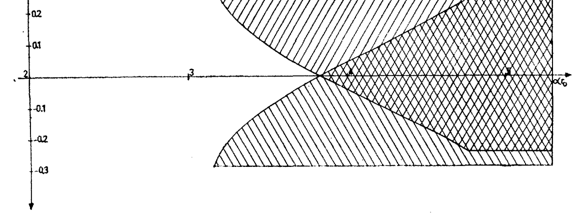

FIG. 2. Instability diagram in terms of k/α and αr_0 for $m = \pm 1$.

The latter fact enables us to extend the proof to the case $\kappa^2 > 1$. Since y_{km} for $c_{2km}=0$ and imaginary argument of the Bessel functions has the only zero $r=0$, it also has only one zero for $c_{2km} \neq 0$. Thus for $\kappa^2 > 1$ instability never occurs.

After knowing that the critical value (32) is not affected by possible instabilities of other intervals than the first one, it still might be possible that they enlarge the unstable κ region found from the first interval. We were able to prove that this is not the case for $\alpha r_0 \leq 6$; furthermore we found much evidence to believe that the situation does not change even for values of $\alpha r_0 > 6$ up to infinity.

It should be noted that it is not very important to know the stability conditions for large values of αr_0 since the corresponding equilibria have little chance to be attained because of earlier instabilities. If one continuously increases the mean current density, the equilibrium becomes unstable when αr_0 reaches the critical value (32), thus not being realizable for values of αr_0 much larger than the critical one.

We see from Fig. 2 that when αr_0 reaches the critical value, instability only occurs to wave numbers $\kappa = \pm 0.272$. Then, with increasing current density the unstable κ region becomes larger and larger. Finally, from $\alpha r_0 = 3.832$ the plasma is unstable to all $|m|=1$ perturbations with $|\kappa| \leq 0.272$.

Because $m = \pm 1$ all unstable perturbations are of the screw or kink type. As can be seen from the example represented in Fig. 3, the r-dependent part $s_{km}(r)$ of the radial component of the perturbation is nearly constant throughout the whole plasma, falling to zero only in the immediate vicinity of the rigid wall. The r-dependent part of the φ component has nearly the same value as $s_{km}(r)$ except near the wall, where it reverses sign and increases very steeply to 280 times this value. The r-dependent part of the z component is zero at the axis, increases slowly, but in the immediate vicinity of the wall it reaches the value 140.

We note that the above-described perturbations differ completely from the localized perturbations which lead to Suydam's condition and against which force-free fields are stable for trivial reasons.

We still must find out whether div$\boldsymbol{\xi}$ can be made to vanish identically by an appropriate choice of the component of $\mathbf{R}$ parallel to $\mathbf{B}$ (corresponding to the remarks at the end of Sec. II). It is clear from the mathematical point of view that this is possible because the infinitely long cylinder does not contain any closed line of force. However, as we wish to consider the cylinder as the limiting case of a very thin torus, the perturbations are

[9] The cases m, k and $-m$, $-k$ are symmetric to each other and have the same stability properties.

subject to the supplementary condition that they must be periodic in the z direction. It can easily be shown for our example that the vanishing of div$\boldsymbol{\xi}$ can be attained [at least for $\boldsymbol{\xi}$'s satisfying conditions (a) and (b) above] by specifying the component of $\boldsymbol{\xi}$ parallel to $\mathbf{B}$ to have the same period $2\pi/k$ in the z direction as the two components perpendicular to $\mathbf{B}$. Thus, in view of the considerations in Sec. II, we find that our stability conditions are valid for all pressures p.

IV. THE INSTABILITY GROWTH RATE

The constraint (8), though yielding much analytical simplification in calculating $\delta W_{1\min}$, does not allow us to calculate the instability growth rates directly as would have been possible with the usual normalization of $\boldsymbol{\xi}$ itself. However, as the growth rate is given by the maximum of the expression

$$\omega=\left(-2\delta W \Big/ \int \rho\xi^2 dv\right)^{1/2} \tag{34}$$

(ρ being the mass density) with respect to all admissible perturbations, we may find a lower limit of the growth rate by calculating (34) for our unstable perturbations.[6] Furthermore, we may expect that this limit is close to the real growth rate since Fig. 1 shows that there is only a narrow region for the unstable perturbations.

Taking values for the parameters estimated to be near the maximum growth rate from Fig. 1 ($r_0=3.4$, $\kappa=0.145$, $b=0.997$), we calculated numerically the expression (34) for a hydrogen plasma. The result is uncertain to a factor of 2 because of some approximation we applied to simplify the calculation of the denominator. We obtained for the growth time $\tau=2\pi/\omega$

$$\tau\approx 3\times 10^{-9}[r_0(n)^{1/2}/B] \text{ sec}, \tag{35}$$

where the radius r_0 has to be taken in cm, the particle density n in cm^{-3}, and the magnetic field strength B in gauss.

In comparison with growth times of pinch instabilities the values (35) are rather large.

Evaluating (35) for the arms of spiral galaxies[10] ($r_0=250$ parsec, $n=1$ cm^{-3}, $B=7\times 10^{-6}$ G) we find $\tau\approx 10^{10}$ years which is of the order of the age of the universe. This value has been calculated without taking into account the gravitational forces. However, following Chandrasekhar and Fermi[11] we assume that the density in the unperturbed state can be treated as roughly constant, so that the gravitational energy is not changed because of $\mathbf{n}\cdot\boldsymbol{\xi}=0$ and div$\boldsymbol{\xi}=0$. Thus, the calculated value for τ gives the right order of magnitude for our special perturbations. Obviously, one must expect still larger growth rates if one allows for a deformation of the boundary.

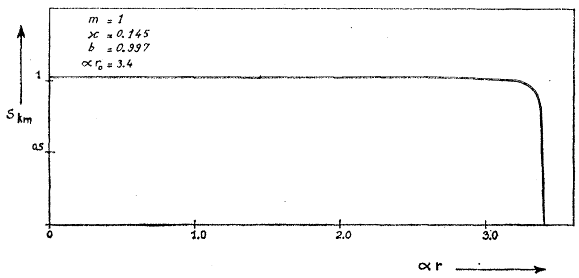

FIG. 3. A plot of s_{km}, the radial variation of the r component of the perturbation. The parameters have been taken to make the perturbation unstable and to be near that of maximum growth rate. Since $m=1$ the perturbation represents a rigid displacement up to about $\alpha r=3$.

The minimum wavelength of our unstable $m=1$ perturbations is smaller than the critical wavelength for gravitational instability (calculated for $m=0$ modes[11]). Thus the presence of force-free fields has, indeed, a destabilizing effect.

The reason that Trehan[2] found a stabilizing effect is due to the fact that he restricts himself to $m=0$ modes. However, the destabilizing effect found in this paper is probably not too important because the growth rates are so small as long as one considers only perturbations which do not move the boundary.

ACKNOWLEDGMENTS

The authors wish to express their deep gratitude to Professor E. Schatzman for stimulating their interest in the stability of force-free fields and for many fruitful discussions. It is also our pleasant duty to thank Dr. Trocheris and Dr. Kulsrud for their interesting and valuable comments. One of us (D. K. C.) thanks Professor Dr. J. L. Verhaeghe for facilities.

[10] The rigid wall model for laboratory plasmas must naturally be replaced by the mathematically equivalent assumption that $\mathbf{n}\cdot\boldsymbol{\xi}=0$ at the boundary. If we take Trehan's (see reference 2) model (specified for $\beta=0$), which prescribes that at the boundary the equilibrium magnetic field must be again parallel to the cylinder axis, we are in the unstable case since all the zeros of $J_1(\alpha r)$ lie beyond our critical value (32).

[11] S. Chandrasekhar and E. Fermi, Astrophys. J. **118**, 116 (1953).