where the notation is that of reference 9. The values of the interaction radii, R_1 and R_2, and the ratio Λ_2/Λ_1 were chosen to provide the best fit to the experimental data.

Figure 4 shows the experimental angular distributions of neutrons leaving the Ne^{20} nucleus in its first excited state. These distributions did not show sufficient character to warrant analysis. They are included for reference only.

Figure 5 shows the neutron yield in the forward direction as a function of deuteron energy from $E_d=0.65$ to 1.3 MeV.

The theoretical angular distributions of neutrons calculated on the basis of the plane wave Born approximation in which exchange effects are included are in good qualitative agreement with the experimental data. Of course, this interpretation of the experimental data is by no means unique. Any plane wave calculation of angular distributions in stripping reactions is subject to the qualification that various effects have been neglected. For a target nucleus as heavy as fluorine, Coulomb effects alone could be significant in explaining one of the principal characteristics of the distribution, i.e., the width of the forward lobe.

However, because of the conflicting results in the literature concerning the gross features of the reaction, it was felt that an effort to fit the smallest details of the angular distribution would be premature. Following the suggestion of Benenson *et al.*[2] an exhaustive but unsuccessful effort was made to fit the experimental data with an entirely different model. A separate plane wave Born approximation calculation was made in which exchange effects were neglected, but the effect of multiple values of proton capture angular momentum was incorporated. The capture angular momentum was given the values of $1_p=0$ and $1_p=2$. The same interaction radius was used in each case, but the magnitude of the interference then was chosen to fit the data. This procedure gave results in conflict with the experimental data. As a result, this approach was abandoned and exchange effects were incorporated in the calculation. The incorporation of the heavy particle stripping mode gave predictions consistent with the experimental observations.

Compound nucleus effects were also neglected in the calculation. The yield curves shown in Fig. 5 seem to indicate that this approximation was justified.

Further experimental work is being carried out at higher energies on both the angular distributions and the neutron gamma-ray correlations corresponding to the first excited state transition.

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Threshold Effect in Elastic Scattering According to the Optical Model*

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An expression is derived for change in the differential elastic scattering cross section caused by the presence of a reaction threshold, in a situation in which it is useful to average over many compound resonances. The derivation is based on the assumptions underlying the optical model and on the usual statistical assumptions made in the evaluation of energy-averaged cross sections. In essence the derivation gives the angular distribution of that part of the fluctuation cross section which is caused by the reaction, and relates this in a simple way to the reaction cross section.

It is shown, although not in a rigorous manner, that for closely-spaced resonances an energy average of the usual Wigner cusp expression, as given by Baz and Newton for example, yields the previously calculated threshold effect on the differential elastic scattering cross sections. Also an expression is given for the threshold effect on the differential elastic scattering cross section in a situation in which the phase shift of each partial wave consists of a part that varies slowly with energy and a part that fluctuates about zero as the energy of the bombarding particle is changed.

For simplicity all calculations are restricted to the case of spin-zero target nucleus and spin-zero and spin-one-half bombarding particle. Furthermore, the derivations assume that only one incident partial wave is dominant in the reaction cross section near threshold. The calculations are applied to recent measurements of Wells, Tucker, and Meyerhof of the differential elastic and inelastic neutron scattering cross sections of cerium in the neighborhood of the threshold for excitation of the 1.60-MeV, 2^+, first excited state of Ce^{140}. It is shown that the computed threshold effect appears to account completely for a marked decrease which had been found in the differential elastic scattering cross section of Ce above 1.60-MeV.

I. INTRODUCTION

RECENT investigation[1] of the elastic differential neutron cross section of cerium has shown a marked decrease in cross section above a neutron energy corresponding to the first excited state of Ce^{140}. Moldauer[2] has interpreted this effect in terms of the optical model and his theory[3] of average neutron reaction cross sections.

The present paper presents a simple calculation of the *change* in the differential elastic cross section caused by

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[1] J. T. Wells, A. B. Tucker, and W. E. Meyerhof (to be published).

[2] P. A. Moldauer (to be published).

[3] P. A. Moldauer, Phys. Rev. 123, 968 (1961).

an inelastic threshold and relates this change to the inelastic cross section in a simple way. The calculation is based on the assumption underlying the optical model and the usual statistical assumptions which accompany a calculation of energy-averaged cross sections. The results are useful if only one incident partial wave (called l' below) is dominant in the inelastic cross section; this usually occurs in the neighborhood of the threshold of an inelastic cross section. For example, in the inelastic neutron scattering leading to the 2^+ first excited state of Ce^{140}, all neutron incident waves besides the d wave contribute less than 10% to the cross section for approximately 150 keV above threshold.[4] This is caused by the large transmission coefficient for the outgoing s-wave neutrons.[5]

In order to avoid complications caused by spin factors, we shall assume that the target has spin zero. We shall treat cases of bombarding particles with spin zero and spin one-half. For simplicity, we shall discuss the case where the inelastic or reaction threshold is the one of lowest energy.

II. BOMBARDING PARTICLE WITH SPIN ZERO

Since we assume the target spin is zero, each diagonal element of the collision matrix needs to be denoted only by the orbital angular momentum l of the incident partial wave and will be called U_l.[6] The differential elastic cross section $d\sigma_e/d\omega$ and the scattering amplitude f are related to U_l by the well-known formula

$$d\sigma_e/d\omega=|f(\theta)|^2,\quad f(\theta)=(i\lambda\!\!\!^{-}/2)\times\sum_l(2l+1)(1-U_l)P_l(\cos\theta),\quad (1)$$

where $\lambda\!\!\!^{-}$ is the c.m. reduced de Broglie wavelength of the bombarding particle, P_l is the Legendre polynomial of order l, and θ is the c.m. scattering angle. The integrated elastic scattering cross section σ_e, the reaction cross section σ_r, and the total cross section σ_t are

$$\sigma_e=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)|1-U_l|^2\equiv\sum_l\sigma_e^{(l)},\quad (2)$$

$$\sigma_r=\pi\lambda\!\!\!^{-2}\sum_l(2l+1)(1-|U_l|^2)\equiv\sum_l\sigma_r^{(l)},\quad (3)$$

$$\sigma_t=2\pi\lambda\!\!\!^{-2}\sum_l(2l+1)(1-\mathrm{Re}U_l)\equiv\sum_l\sigma_t^{(l)}.\quad (4)$$

We now assume that in any finite interval of energy of the bombarding particle there are sufficiently many resonances so that it is meaningful to define any energy-averaged scattering matrix element $\bar{U}_l$. The actual scattering matrix element can then be given by

$$U_l=\bar{U}_l+\Delta U_l,\quad (5)$$

where

$$\langle\Delta U_l\rangle_{\mathrm{av}}=0;\quad (6)$$

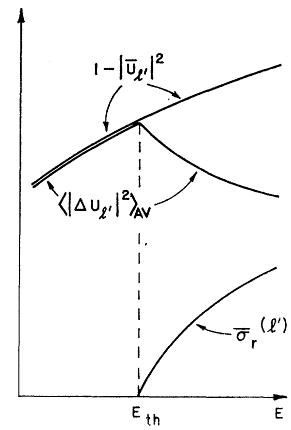

FIG. 1. Schematic representation of energy variation of the optical-model transmission coefficient $1-|\bar{U}_{l'}|^2$, of the fluctuation cross-section coefficient $\langle|\Delta U_{l'}|^2\rangle_{\mathrm{av}}$ and of reaction cross section $\bar{\sigma}_r^{(l')}$. Below the threshold energy E_{th}, the first two curves have been separated for clarity.

in our notation a bar over the symbol and the brackets $\langle\ \rangle_{\mathrm{av}}$ both denote energy averages. If we denote optical-model quantities by $<\ >_{\mathrm{opt}}$ the basic assumption of the optical model is[7]

$$<U_l>_{\mathrm{opt}}=\bar{U}_l.\quad (7)$$

It follows from this that according to the optical model, $\bar{U}_l$ is a smoothly varying function of the bombarding energy E, even across inelastic thresholds. Hence, from expressions (1) and (4) one sees that $\bar{f}=<f>_{\mathrm{opt}}$ and $\bar{\sigma}_t=<\sigma_t>_{\mathrm{opt}}$ are also smooth functions of energy across inelastic thresholds:

$$\bar{f}=(i\lambda\!\!\!^{-}/2)\sum_l(2l+1)(1-\bar{U}_l)P_l,\quad (8)$$

$$\bar{\sigma}_t^{(l)}=2\pi\lambda\!\!\!^{-2}(2l+1)(1-\mathrm{Re}\bar{U}_l).\quad (9)$$

Furthermore, using (5) and (6) one finds the well-known expressions[7]

$$\bar{\sigma}_e^{(l)}=\pi\lambda\!\!\!^{-2}(2l+1)(|1-\bar{U}_l|^2+\langle|\Delta U_l|^2\rangle_{\mathrm{av}}),\quad (10)$$

$$\bar{\sigma}_r^{(l)}=\pi\lambda\!\!\!^{-2}(2l+1)(1-|\bar{U}_l|^2-\langle|\Delta U_l|^2\rangle_{\mathrm{av}}).\quad (11)$$

The terms proportional to $\langle|\Delta U_l|^2\rangle_{\mathrm{av}}$ are the partial-wave fluctuation cross sections. It should be noted that $\bar{\sigma}_r$ is the measured reaction cross section, whereas $<\sigma_r>_{\mathrm{opt}}$ would be the (compound nucleus) absorption cross section.[7]

We see from (11) that below any reaction threshold $\langle|\Delta U_l|^2\rangle_{\mathrm{av}}=1-|\bar{U}_l|^2$. If we assume that at a certain threshold energy E_{th} a reaction takes place only in the (l')th partial wave, there must be a "deviation from smoothness" in $\langle|\Delta U_{l'}|^2\rangle_{\mathrm{av}}$ at $E=E_{\mathrm{th}}$, since $1-|\bar{U}_{l'}|^2$ is a smooth function of energy. This situation is shown schematically in Fig. 1 and is the main physical content of this paper. It is well known that the experimental scattering cross section will not be absolutely smooth at any energy because of fluctuations of the widths of overlapping resonances as well as interference effects.[8]

[4] P. A. Moldauer (private communication).

[5] Sufficiently close to threshold, inelastic neutron scattering is always dominated by that incident partial wave which accompanies an outgoing s wave.

[6] This is identical to η_l of Blatt and Weisskopf [J. M. Blatt and W. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952)].

[7] H. Feshbach, C. E. Porter, and V. F. Weisskopf, Phys. Rev. **96**, 448 (1954).

[8] See, for example, T. Ericson, Phys. Rev. Letters **5**, 430 (1960) and references given there. Also T. Ericson, "The Compound

The "deviation from smoothness" across a reaction threshold, which we consider here, is on a larger energy scale—say typically over 100 keV, whereas the fluctuations in the cross section may have typical widths of 10 keV or less.[8] We are now ready to consider the energy-averaged differential elastic scattering cross section

$$\langle d\sigma_e/d\omega\rangle_{\rm av}=|\bar{f}|^2+\langle|\Delta f|^2\rangle_{\rm av}, \quad (12)$$

where

$$\Delta f=-(i\lambda\!\!\!^{-}/2)\textstyle\sum_l(2l+1)\Delta U_l P_l. \quad (13)$$

Since we are interested only in the "deviation from smoothness" across the inelastic threshold of $\langle d\sigma_e/d\omega\rangle_{\rm av}$, which we shall call $\Delta\langle d\sigma_e/d\omega\rangle_{\rm av}$, we shall not consider $|\bar{f}|^2$ any further and in $\langle|f|^2\rangle_{\rm av}$ separate out the (l')th partial wave

$$\langle|\Delta f|^2\rangle_{\rm av}=(\lambda\!\!\!^{-2}/4)\{\sum_{l\neq l'}(2l+1)^2\langle|\Delta U_l|^2\rangle_{\rm av}(P_l)^2 +2[\sum_{l\neq l'}(2l+1)(2l'+1)\,{\rm Re}\langle\Delta U_l^*\Delta U_{l'}\rangle_{\rm av}P_lP_{l'}] +(2l'+1)^2\langle|\Delta U_{l'}|^2\rangle_{\rm av}(P_{l'})^2\}. \quad (14)$$

If we now make the usual statistical assumption that the terms ${\rm Re}\langle\Delta U_l^*\Delta U_{l'}\rangle_{\rm av}$ average out to zero, since the fluctuations ΔU of different partial waves are uncorrelated, we obtain an expression for the differential fluctuation cross section analogous to one derived by Feshbach.[9] For our purposes, we shall substitute $\langle|\Delta U_l|^2\rangle_{\rm av}=1-|\bar{U}_l|^2$ for all those channels in which no reaction takes place and add and subtract the quantity $(2l'+1)^2(1-|\bar{U}_{l'}|^2)(P_{l'})^2$ to obtain, with the help of expression (11),

$$\langle|\Delta f|^2\rangle_{\rm av}=(\lambda\!\!\!^{-2}/4)[\textstyle\sum_l(2l+1)^2(1-|\bar{U}_l|^2)(P_l)^2 -(2l'+1)(P_{l'})^2\bar{\sigma}_r^{(l')}/(\pi\lambda\!\!\!^{-2})]. \quad (15)$$

The "deviation from a smooth energy dependence" across the inelastic threshold is given by the second term

$$\Delta\langle d\sigma_e/d\omega\rangle_{\rm av}=-(2l'+1)(P_{l'})^2\bar{\sigma}_r^{(l')}/(4\pi). \quad (16)$$

Therefore, above an inelastic threshold the energy-averaged differential elastic scattering cross section always decreases. Also, there is, according to the present model, no effect of the reaction below threshold. Both these features differ from the Wigner cusp phenomenon[10–13] in the (unaveraged) differential elastic scattering cross section, as we shall discuss below. We can integrate expression (16) over all angles to obtain the deviation from smooth energy dependence of the integrated elastic cross section and find the obvious result.

$$\Delta\bar{\sigma}_e=-\bar{\sigma}_r^{(l')}, \quad (17)$$

which could have been obtained directly from $\bar{\sigma}_e=\bar{\sigma}_t-\bar{\sigma}_r$. We might note that if more than one incident partial wave contributes to $\bar{\sigma}_r$, the equation corresponding to (16) is easily obtained, but is no longer so simple.

III. BOMBARDING PARTICLE WITH SPIN ONE-HALF

The above method can be extended immediately to any other spin situation, but we shall give the results only for the case of target spin zero and bombarding particle spin one-half, because this is of immediate interest.

For any incident partial wave l, the compound system can have total angular momentum $l+\frac{1}{2}$ and $l-\frac{1}{2}$, and we denote these cases by $+$ and $-$, respectively. If we label the equations analogous to each of the formulas in the target-spin zero case by primes, we have the well-known expressions [$P_l^{(1)}(\cos\theta)$ = associated Legendre polynomial of order 1]

$$d\sigma_e/d\omega=|f_0(\theta)|^2+|f_1(\theta)|^2,$$

$$f_0(\theta)=(i\lambda\!\!\!^{-}/2)\textstyle\sum_l[(l+1)(1-U_{l+})+l(1-U_{l-})]P_l(\cos\theta),$$

$$f_1(\theta)=-(i\lambda\!\!\!^{-}/2)\textstyle\sum_l(U_{l+}-U_{l-})P_l^{(1)}(\cos\theta), \quad (1')$$

$$\sigma_e=\pi\lambda\!\!\!^{-2}\textstyle\sum_l[(l+1)|1-U_{l+}|^2+l|1-U_{l-}|^2]\equiv\sum_l\sigma_e^{(l)}, \quad (2')$$

$$\sigma_r=\pi\lambda\!\!\!^{-2}\textstyle\sum_l[(l+1)(1-|U_{l+}|^2)+l(1-|U_{l-}|^2)]\equiv\sum_l\sigma_r^{(l)}, \quad (3')$$

$$\sigma_t=2\pi\lambda\!\!\!^{-2}\textstyle\sum_l[(l+1)(1-{\rm Re}U_{l+})+l(1-{\rm Re}U_{l-})]\equiv\sum_l\sigma_t^{(l)}. \quad (4')$$

We assume that Eqs. (5) and (6) apply to U_{l+} and U_{l-} separately; in this case spin-orbit effects have to be included in the optical model and

$$<U_{l+}>_{\rm opt}=\bar{U}_{l+},\quad <U_{l-}>_{\rm opt}=\bar{U}_{l-} \quad (7')$$

are again smooth functions of the energy according to the optical model. All expressions similar to those up to Eq. (14) follow immediately. We give only the important ones

$$\bar{f}_0=(i\lambda\!\!\!^{-}/2)\textstyle\sum_l[(l+1)(1-\bar{U}_{l+})+l(1-\bar{U}_{l-})]P_l,$$

$$\bar{f}_1=-(i\lambda\!\!\!^{-}/2)\textstyle\sum_l(\bar{U}_{l+}-\bar{U}_{l-})P_l^{(1)}, \quad (8')$$

$$\bar{\sigma}_r^{(l)}=\pi\lambda\!\!\!^{-2}[(l+1)(1-|\bar{U}_{l+}|^2-\langle|\Delta U_{l+}|^2\rangle_{\rm av})+l(1-|\bar{U}_{l-}|^2-\langle|\Delta U_{l-}|^2\rangle_{\rm av})], \quad (11')$$

$$\equiv\pi\lambda\!\!\!^{-2}[(l+1)Q_{l+}+lQ_{l-}], \quad (18)$$

where Q_{l+} and Q_{l-} have been defined to simplify the notation.

$$\langle d\sigma_e/d\omega\rangle_{\rm av}=|\bar{f}_0|^2+|\bar{f}_1|^2+\langle|\Delta f_0|^2\rangle_{\rm av}+\langle|\Delta f_1|^2\rangle_{\rm av}, \quad (12')$$

Nucleus and the Random Phase Approximation" (unpublished lectures, Varenna, Italy, August 7–16, 1961); R. Stevens and D. Brink (unpublished).

[9] H. Feshbach, in *Nuclear Spectroscopy*, edited by F. Ajzenberg-Selove (Academic Press Inc., New York, 1960), Part B, p. 1033.

[10] E. P. Wigner, Phys. Rev. **73**, 1002 (1948).

[11] A. I. Baz, Soviet Phys.—JETP **6**, 709 (1958). Equation (3.2) in this paper should not have the factor $(2l'+1)$. Also in Eq. (4.4), in h, the sign preceding a_2 should be $-$.

[12] R. G. Newton, Phys. Rev. **114**, 1611 (1959).

[13] L. Fonda, Nuovo cimento **20**, 116 (1961) and references given there.

$$\Delta f_0 = -(i\lambda\!\!\!^{-}/2)\sum_l [(l+1)\Delta U_{l+} + l\Delta U_{l-}]P_l,$$
$$\Delta f_1 = -(i\lambda\!\!\!^{-}/2)\sum_l (\Delta U_{l+} - \Delta U_{l-})P_l^{(1)}. \qquad (13')$$

Again $|\bar{f}_0|^2 + |\bar{f}_1|^2$ varies smoothly across an inelastic threshold and is not of further interest here. Separating out the (l')th incident partial wave,

$$\begin{aligned}\langle|\Delta f_0|^2\rangle_{\rm av} + \langle|\Delta f_1|^2\rangle_{\rm av} &= (\lambda\!\!\!^{-2}/4)\{\sum_{l\neq l'}([(l+1)^2\langle|\Delta U_{l+}|^2\rangle_{\rm av} \\ &+ l^2\langle|\Delta U_{l-}|^2\rangle_{\rm av}](P_l)^2 + (\langle|\Delta U_{l+}|^2\rangle_{\rm av} \\ &+ \langle|\Delta U_{l-}|^2\rangle_{\rm av})(P_l^{(1)})^2) + [(l'+1)^2\langle|\Delta U_{l'+}|^2\rangle_{\rm av} \\ &+ l'^2\langle|\Delta U_{l'-}|^2\rangle_{\rm av}](P_{l'})^2 + (\langle|\Delta U_{l'+}|^2\rangle_{\rm av} \\ &+ \langle|\Delta U_{l'-}|^2\rangle_{\rm av})(P_{l'}^{(1)})^2\}, \qquad (14')\end{aligned}$$

where we have made the statistical assumption that terms of the form

$${\rm Re}\langle\Delta U_{l\pm}{}^*\Delta U_{l\mp}\rangle_{\rm av} \quad \text{and} \quad {\rm Re}\langle\Delta U_{l(+\text{ or }-)}{}^*\Delta U_{l'(+\text{ or }-)}\rangle_{\rm av}$$

average to zero because the phase shifts of the various channel spins are uncorrelated. Making substitutions similar to those preceding Eq. (15) and using the notation introduced by Eq. (18) we obtain

$$\begin{aligned}\langle|\Delta f_0|^2\rangle_{\rm av} + \langle|\Delta f_1|^2\rangle_{\rm av} &= (\lambda\!\!\!^{-2}/4)\{\sum_l([(l+1)^2(1-|\bar{U}_{l+}|^2) \\ &+ l^2(1-|\bar{U}_{l-}|^2)](P_l)^2 \\ &+ [(1-|\bar{U}_{l+}|^2) + (1-|\bar{U}_{l-}|^2)](P_l^{(1)})^2) \\ &- [(l'+1)^2Q_{l'+} + l'^2Q_{l'-}](P_{l'})^2 \\ &- [Q_{l'+} + Q_{l'-}](P_{l'}^{(1)})^2\}. \qquad (15')\end{aligned}$$

The deviation from smooth energy dependence across the reaction threshold is given by the terms containing $Q_{l'+}$ and $Q_{l'-}$ and with Eq. (19) we find

$$\Delta\langle d\sigma_e/d\omega\rangle_{\rm av} = -\left[\frac{(l'+1)^2 + l'^2\bar{r}_{l'}}{l'+1+l'\bar{r}_{l'}}(P_{l'})^2 + \frac{1+\bar{r}_{l'}}{l'+1+l'\bar{r}_{l'}}(P_{l'}^{(1)})^2\right]\frac{\bar{\sigma}_r^{(l')}}{4\pi}, \qquad (16')$$

where

$$\bar{r}_{l'} = Q_{l'-}/Q_{l'+}. \qquad (19)$$

It turns out that expression (16′) is only weakly dependent on $\bar{r}_{l'}$ and therefore we can make the reasonable assumption that $\bar{r}_{l'} \cong 1$, so that

$$\Delta\langle d\sigma_e/d\omega\rangle_{\rm av} \cong -\left[\frac{(l'+1)^2 + l'^2}{2l'+1}(P_{l'})^2 + \frac{2}{2l'+1}(P_{l'}^{(1)})^2\right]\frac{\bar{\sigma}_r^{(l')}}{4\pi}. \qquad (20)$$

As a check on these expressions one can integrate Eq. (16′) over all angles and one finds Eq. (17).

IV. RELATION TO WIGNER CUSP

Before discussing our recent experiments[1] with reference to Eq. (20) we show below equations similar to those derived by Baz[11] and Newton[12] for threshold effects in various cross sections under the assumptions of analytic continuation of U_l across the reaction threshold and slowly varying phase shift.[10,13] Denoting again by $\Delta\sigma$ the deviation of σ caused by the reaction threshold and restricting ourselves to spin-zero targets and a single reaction threshold in the (l')th incident partial wave, we can put the expressions into the following forms:

$$\Delta d\sigma_e/d\omega = -\frac{[(d\sigma_e/d\omega)_{E_{\rm th}}]^{1/2}}{\lambda\!\!\!^{-}}P_{l'}\frac{\sigma_r^{(l')}(|E-E_{\rm th}|, l_0)}{2\pi}\times\begin{cases}\sin(2\delta_{l'}-\alpha), & E>E_{\rm th}\\ (-1)^{l_0}\cos(2\delta_{l'}-\alpha), & E<E_{\rm th}\end{cases} \qquad (21)$$

$$\Delta\sigma_e = -\sigma_r^{(l')}(|E-E_{\rm th}|, l_0)\times\begin{cases}2\sin^2\delta_{l'} & E>E_{\rm th}\\ (-1)^{l_0}\sin 2\delta_{l'}, & E<E_{\rm th}\end{cases} \qquad (22)$$

$$\Delta\sigma_t = \sigma_r^{(l')}(|E-E_{\rm th}|, l_0)\times\begin{cases}\cos 2\delta_{l'} & E>E_{\rm th}\\ (-1)^{l_0+1}\sin 2\delta_{l'}, & E<E_{\rm th}.\end{cases} \qquad (23)$$

These expressions assume that

$$\sigma_r^{(l')}(|E-E_{\rm th}|, l_0) = \pi\lambda\!\!\!^{-2}(2l'+1)A_{l'l_0}|k_0|^{2l_0+1}, \qquad (24)$$

where l_0 is the angular momentum of the outgoing particle, k_0 its c.m. wave number

$$k_0 = (2M/\hbar^2)^{1/2}(E-E_{\rm th})^{1/2}$$

and $A_{l'l_0}$ is a constant.[11] M is the reduced mass in the outgoing channel, $\delta_{l'}$ is the phase shift at threshold $[U_{l'}(E_{\rm th}) = e^{i2\delta_{l'}(E_{\rm th})}]$, and α is the phase of f at threshold $[f(\theta) = e^{i\alpha(\theta)}(d\sigma_e/d\omega)^{1/2}]$. The above expressions are valid for any outgoing angular momentum l_0 but give rise to a real cusp only if $l_0=0$ and if there are no Coulomb effects in the outgoing channel.[13,14] As has been pointed out before,[13] σ_e can only decrease above threshold. Also, the below-threshold effect on σ_e and σ_t

[14] G. Breit, Phys. Rev. **107**, 1612 (1957) and *Encyclopedia of Physics*, edited by S. Flügge (Springer-Verlag, Berlin, Germany, 1959), Vol. 41, Part I, p. 274 ff.

is the same, as it should be since there is no reaction below threshold.

For spin-one-half bombarding particles we restrict ourselves to the case of zero outgoing angular momentum ($l_0=0$) and set

$$\sigma_r^{(l')}=\pi\lambdabar^2[(l'+1)A_{l'+}+l'A_{l'-}]k_0,$$

where $A_{l'+}$ and $A_{l'-}$ are energy independent factors depending on l' and on the spin $l'\pm\frac{1}{2}$ of the compound system.[15] Under these conditions

$$\Delta\sigma_e=-\sigma_r^{(l')}(|E-E_{\rm th}|)\times\begin{cases}\dfrac{2[(l'+1)\sin^2\delta_{l'+}+l'r_{l'}\sin^2\delta_{l'-}]}{l'+1+l'r_{l'}}, & E>E_{\rm th}\\[2ex] \dfrac{(l'+1)\sin2\delta_{l'+}+l'r_{l'}\sin2\delta_{l'-}}{l'+1+l'r_{l'}}, & E<E_{\rm th}\end{cases}\tag{25}$$

$$\Delta\sigma_t=\sigma_r^{(l')}(|E-E_{\rm th}|)\times\begin{cases}\dfrac{(l'+1)\cos2\delta_{l'+}+l'r_{l'}\cos2\delta_{l'-}}{l'+1+l'r_{l'}}, & E>E_{\rm th}\\[2ex] \dfrac{-[(l'+1)\sin2\delta_{l'+}+l'r_{l'}\sin2\delta_{l'-}]}{l'+1+l'r_{l'}}, & E<E_{\rm th}\end{cases}\tag{26}$$

where $r_{l'}\equiv A_{l'-}/A_{l'+}$. If $\delta_{l'+}=\delta_{l'-}$ these formulas reduce to Eqs. (22) and (23), respectively, for $l_0=0$.

For the cusp effect in the differential elastic cross section one finds[11] (for target spin zero) in the case of spin-one-half bombarding particles

$$\begin{aligned}\Delta d\sigma_e/d\omega=-(\lambdabar/2)\{|f_0|[(l'+1)A_{l'+}\sin(2\delta_{l'+}-\alpha_0)\\ +l'A_{l'-}\sin(2\delta_{l'-}-\alpha_0)]P_{l'}\\ +|f_1|[A_{l'+}\sin(2\delta_{l'+}-\alpha_1)\\ -A_{l'-}\sin(2\delta_{l'-}-\alpha_1)]P_{l'}^{(1)}\}k_0,\quad E>E_{\rm th}\end{aligned}\tag{27}$$

where f_0 and f_1 have been defined in Eq. (1′) and each α is defined by $f=|f|e^{i\alpha}$. For $E<E_{\rm th}$ one has to substitute in Eq. (27) $k_0\rightarrow i|k_0|$ and $\sin\rightarrow\cos$. This formula reduces to Eq. (21) (with $l_0=0$) only if $A_{l+}=A_{l-}$ and $\delta_{l+}=\delta_{l-}$ for all l.

It is of interest to relate these formulas for the Wigner cusp effect to the threshold effects in energy-averaged cross sections. Although the expansion of the matrix element $U_{l'}$ around threshold, which leads to the expressions (21) to (26), is usually derived[11,12] with each value of $\delta_{l'}$ taken exactly at threshold, we can assume that the form of $U_{l'}$ as a function of energy persists even away from threshold [see Eqs. (29) or (32) below and footnote 16]. Under these conditions, if we average over an energy interval in which many resonances exist, we can expect [see Eqs. (22), (23), (25), (26)] that $\delta_{l'}$ may vary as a function of energy so that

$$\begin{aligned}\langle\sigma_r^{(l')}\sin2\delta_{l'}\rangle_{\rm av}\cong0,\quad \langle\sigma_r^{(l')}\cos2\delta_{l'}\rangle_{\rm av}\cong0,\\ \langle\sigma_r^{(l')}\sin^2\delta_{l'}\rangle_{\rm av}\cong\bar\sigma_r^{(l')}/2.\end{aligned}\tag{28}$$

The first of these assumptions is related to the random-sign approximation,[8] but certainly all assumptions require detailed examination on the basis of resonance theory which we shall make in a future publication. It is easily shown[2,4] that for narrow isolated resonances the last assumption (28) is indeed verified. The word "narrow" implies that the width of each resonance must be very small compared to the energy interval $\hbar^2/(2Ma^2)$, where a is the reaction channel radius. The energy intervals over which expressions (28) are averaged must also be small compared to $\hbar^2/(2Ma^2)$, but must contain many compound resonances. We can assume that the expressions (28) apply also to $\delta_{l'+}$ and $\delta_{l'-}$. With assumptions (28) we find

$$\begin{aligned}\Delta\bar\sigma_e&\cong-\bar\sigma_r^{(l')}, && E>E_{\rm th}\\ &\cong0, && E<E_{\rm th}\\ \Delta\bar\sigma_t&\cong0, && E>E_{\rm th}\ \text{and}\ E<E_{\rm th}\end{aligned}$$

which are the optical-model expressions [see discussion preceding Eq. (8) and see Eq. (17)].

One might be misled in believing that on the basis of Eq. (28) Eqs. (21) and (27) yield $\Delta\langle d\sigma_e/d\omega\rangle_{\rm av}\cong0$, but the term $(d\sigma_e/d\omega)^{1/2}$ contains the phase $\delta_{l'}$ and hence a correlation exists which must be examined by going back one step in the derivation of these equations. We shall discuss only the spinless case, for which the expansion of $U_{l'}$ gives close to threshold[11,12,16] (for $l_0=0$)

[15] In terms of the notation of Baz (reference 11), $A_-=2a_1R$, $A_+=2a_2R$; it should be noted that Eqs. (21) to (23) assume that no even powers of $|k_0|$ are separated out from the phase shift $\delta_{l'}$ near threshold. As pointed out by Breit, even powers of $|k_0|$ give a smooth energy dependence across threshold of the cross section or of its derivatives with respect to energy.

[16] It is easily shown that Eqs. (29) or (32) with $\delta_{l'}=\delta_{l'}(E)$ are indeed a very good approximation above threshold and that $2\delta_{l'}$ is the complete phase of $U_{l'}$. This follows from Eq. (3) which shows that

$$|U_{l'}|=\{1-\sigma_r^{(l')}/[(2l'+1)\pi\lambdabar^2]\}^{1/2},$$

and hence

$$\begin{aligned}U_{l'}&=e^{2i\delta_{l'}}\{1-\sigma_r^{(l')}/[(2l'+1)\pi\lambdabar^2]\}^{1/2}\\ &\cong e^{2i\delta_{l'}}\{1-\tfrac{1}{2}\sigma_r^{(l')}/[(2l'+1)\pi\lambdabar^2]\},\end{aligned}$$

which is just Eq. (32) with substitution of Eq. (24). The below threshold dependence of Eq. (32) needs a special derivation. One would expect that analytic continuation below threshold of the expressions for $U_{l'}$ just given with $k_0\rightarrow i|k_0|$ should be valid. In this case, though, $2\delta_{l'}$ is obviously not the complete phase of $U_{l'}$, but for the averaging process we require only a sufficient dependence on E ($<E_{\rm th}$) that the assumed fluctuations of $\delta_{l'}$ as a function

$$U_{l'}=e^{2i\delta_{l'}}(1-\tfrac{1}{2}A_{l'}k_0) \quad \text{for} \quad E>E_{\text{th}};$$
$$k_0\to i|k_0| \quad \text{for} \quad E<E_{\text{th}}, \qquad (29)$$

where $A_{l'}$ is energy independent. This expression is valid only if there are no Coulomb effects in the outgoing channel.[13,14] The corresponding reaction cross section is

$$\sigma_r^{(l')}=\pi\lambda\!\!\!^{-2}(2l'+1)A_{l'}k_0. \qquad (30)$$

Equations (29) and (30) assume $A_{l'}k_0\ll 1$. Substitution of Eq. (29) into Eq. (1) gives for the cusp term

$$\Delta d\sigma_e/d\omega=2\,\mathrm{Re}[i(\lambda\!\!\!^{-}/2)\textstyle\sum_l(2l+1)(1-e^{2i\delta_l})P_l]^* \times[i(\lambda\!\!\!^{-}/2)(2l'+1)e^{2i\delta_{l'}}(A_{l'}/2)k_0P_{l'}]$$

for $E>E_{\text{th}}$ and $k_0\to i|k_0|$ for $E<E_{\text{th}}$.

If we make the assumption (28) we see that the only nonzero term in the energy-averaged cross section results from the term $l=l'$ in the sum. Hence

$$\Delta\langle d\sigma_e/d\omega\rangle_{\text{av}}\cong-(\lambda\!\!\!^{-2}/4)(2l'+1)^2A_{l'}(P_{l'})^2\times\begin{cases}\mathrm{Re}k_0, & E>E_{\text{th}}\\ \mathrm{Re}i|k_0|, & E<E_{\text{th}}\end{cases}$$
$$\cong\begin{cases}-(2l'+1)(P_{l'})^2\sigma_r^{(l')}/(4\pi), & E>E_{\text{th}}\\ 0, & E<E_{\text{th}}.\end{cases} \qquad (31)$$

This is just the optical-model result given by Eq. (16) since here $\sigma_r^{(l')}=\bar{\sigma}_r^{(l')}$ [see Eq. (30)]. Although this has been shown only for the case $l_0=0$, it is equally well demonstrated for the case of nonzero l_0, for which near threshold[11,13,16]

$$U_{l'}=e^{2i\delta_{l'}}(1-\tfrac{1}{2}A_{l'l_0}k_0^{2l_0+1}) \quad \text{for} \quad E>E_{\text{th}};$$
$$k_0\to i|k_0| \quad \text{for} \quad E<E_{\text{th}}, \qquad (32)$$

where $A_{l'l_0}$ was introduced in Eq. (24) and is an energy-independent quantity. One can show also that an energy average of Eq. (27) leads to Eq. (16′).

V. COMPARISON WITH EXPERIMENT

Recently the elastic[1] and inelastic[17] scattering of neutrons on Ce has been measured in the neighborhood of the threshold for excitation of the first excited state of Ce^{140}. In order to compare these measurements with Eq. (16′) or (20) one would have to know the smooth energy dependence (i.e., optical-model calculation without threshold effects) of the differential elastic cross section. Since there are fluctuations in the actual differential cross sections,[8] it would be difficult to match the optical-model cross section properly to obtain $\Delta\langle d\sigma_e/d\omega\rangle_{\text{av}}$ by subtraction. Hence, we test our theory in a different way. We add the quantity $\Delta\langle d\sigma_e/d\omega\rangle_{\text{av}}$ (calculated from the measured[17] inelastic cross section) to the experimental[1] cross section. If the theory is correct, the net result should show a "smooth" energy dependence across the inelastic threshold and should follow the trend of optical model calculations (with no threshold effects included). We expect, of course, that the so-computed "smooth" cross section shows the same fluctuations above threshold as the experimental cross section shows below threshold.

Figures 2 and 3 present the experimental results[1,17] in this manner. The left side (a) of each figure gives the experimental differential elastic scattering, cross sections at various angles. The cross sections have been averaged over 40-keV intervals and are shown as histograms. The right side (b) gives the computed cross sections obtained by adding expression (20) to each curve on side (a). It turns out in this case that any reasonable

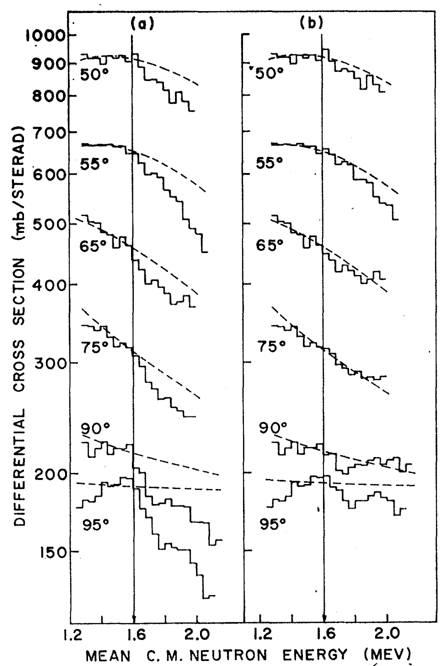

FIG. 2. Left side (a) shows the experimental energy-averaged differential elastic neutron scattering cross section of Ce. Energy averages have been taken over 40 keV; experimental results are from reference 1. The dashed lines are optical-model trends from references 18 and 19 (see text). The threshold effect due to the excitation of the 1.60-MeV first excited state of Ce^{140} is clearly apparent, especially at angles greater than 60°. Right side (b) shows experimental cross section to which calculated threshold effect, Eq. (20), has been added. The resultant curve is expected to follow the optical-model trend both below and above threshold and indeed does at most angles.

of E are not wiped out. J. D. Walecka (private communication) has shown that this is a reasonable expectation. We might also note that if the above expressions for $U_{l'}$ are used in the derivation of Eq. (31) one finds that assumptions (28) lead to expression (31) with $\sigma_r^{(l')}$ replaced by $\bar{\sigma}_r^{(l')}$.

[17] A. B. Tucker, J. T. Wells, and W. E. Meyerhof (to be published).

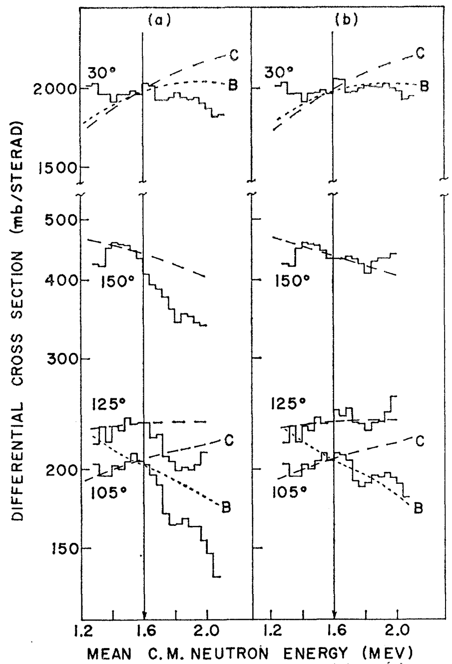

FIG. 3. Same caption applies as to Fig. 2. To demonstrate the sensitivity of optical-model trends to optical-model parameters, the trend (B) predicted by reference 19 has been shown at 30° and 105° in addition to the trend (C) predicted by reference 18.

choice of $\bar{r}_{l'}$ [Eq. (19)], say between 0.5 and 2.0, gives practically the same result as $\bar{r}_{l'}=1.0$. Even the extreme substitutions of 0 and ∞ for $\bar{r}_{l'}$ do not alter the trend of curves (b) significantly. Figure 4 gives the experimental[17] inelastic cross section of Ce^{140}, shown as a histogram for easier comparison with Figs. 2 and 3.

Superimposed on each curve is a smooth dashed line indicating the trend of the optical model. We have used the calculations of Campbell *et al.*[18] for Ce, interpolated in energy for the forward angles by Beyster's[19] computations. For angles greater than 75° the two calculations differ appreciably (probably because of different assumptions about the fluctuation cross section) and we have used Campbell's results. At each angle we have matched the magnitudes of these calculated curves with the experimental results in the energy interval 1.5–1.6 MeV. This may appear somewhat arbitrary, but the errors in the experimental cross sections[1,17] were estimated to be roughly $\pm 15\%$ and only small shifts ($<20\%$) were necessary to match Campbell's predictions at most angles.

It is clear from Figs. 2 and 3, side (a), that there is a marked drop in the cross section above 1.6 MeV, which is especially noticeable at the back angles. At 1.60 MeV Ce^{140} has a first excited state[20] of spin 2^+. Moldauer[2,4] has shown that the inelastic neutron excitation of this state proceeds mainly by the entering d wave for more than one hundred keV above threshold. (This is due to the large s-wave transmission coefficient for the outgoing wave, i.e., $l_0=0$ is predominant.) Hence, we have substituted $l'=2$ in Eq. (20) and used the results of Fig. 4 to obtain the curves (b) in Figs. 2 and 3. One sees that curves (b) follow the optical-model trends remarkably well and that to this accuracy at least the validity of Eq. (20) seems established for Ce. Slight adjustment of optical-model parameters would no doubt change the slopes of some of the optical-model curves. This is shown by the 30° and 105° curves on Fig. 3 in which we give the trend indicated by Beyster's calculations[19] (B) in addition to that indicated by Campbell (C).[18] Also Moldauer[2] has been able to fit the elastic and inelastic scattering results[1,17] in remarkable detail.

VI. DISCUSSION

We have obtained an expression for the deviation from smooth energy dependence across a reaction threshold of the energy-averaged differential elastic scattering cross section in some special situations. We have, on the one hand, made it plausible that this deviation is essentially the energy-averaged Wigner cusp phenome-

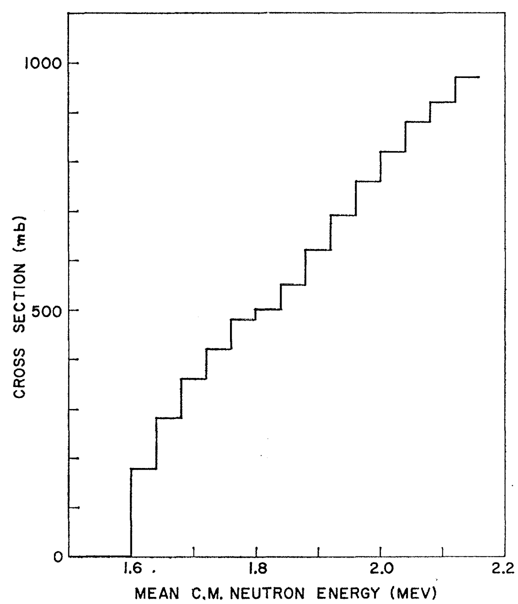

FIG. 4. Inelastic neutron scattering cross section of Ce^{140}. For easier comparison with Figs. 2 and 3 the results of reference 17 are shown as averages over 40-keV neutron energy intervals. The cross section indicates the excitation of the first excited state of Ce^{140} at 1.60 MeV. The cross section shown is per atom of Ce. To obtain the isotopic cross section for Ce^{140}, the ordinate has to be divided by 0.88.

[18] E. J. Campbell, H. Feshbach, C. E. Porter, and V. F. Weisskopf, Massachusetts Institute of Technology Report No. 73, 1960 (unpublished).

[19] J. R. Beyster, Los Alamos Scientific Laboratory Report LA-2099, 1957 (unpublished).

[20] See, e.g., *Nuclear Data Sheets*, edited by K. Way *et al.* (National Academy of Sciences, National Research Council, Washington, D. C., 1961), No. 59-1-84.

non[10–14] and, on the other hand, shown that it is consistent with experiment[1,17] in a region where one expects energy averaging to be valid.

It is clear that the particular variations with energy of the phase shift $\delta_{l'}(E)$, which are implicit in the energy-averaging process of Eq. (6), are an assumption and that there may exist situations in which it might be appropriate to put

$$\delta_{l'}=\bar{\delta}_{l'}+\Delta\delta_{l'}, \quad \text{with} \quad \langle\Delta\delta_{l'}\rangle_{\text{av}}=0, \tag{33}$$

in analogy with Eq. (5); i.e., $\bar{\delta}_{l'}$ is slowly varying with energy in any energy interval of interest. Using Eq. (32) for $U_{l'}$ and writing

$$e^{2i\delta_{l'}}=e^{2i\bar{\delta}_{l'}}(\cos 2\Delta\delta_{l'}+i\sin 2\Delta\delta_{l'}),$$

it is easily shown that for the spin-zero case (bombarding particle and target)

$$\Delta\langle d\sigma_e/d\omega\rangle_{\text{av}}=-(F/2\pi\lambda\!\!\!^{-})P_{l'}\langle\sigma_r^{(l')}(|E-E_{\text{th}}|,l_0)\cos 2\Delta\delta_{l'}\rangle_{\text{av}}\times\begin{cases}\sin(2\bar{\delta}_{l'}-\beta)\\(-1)^{l_0}\cos(2\bar{\delta}_{l'}-\beta)\end{cases}$$
$$-\begin{cases}(2l'+1)(\bar{\sigma}_r^{(l')}/4\pi)(P_{l'})^2, & E>E_{\text{th}}\\0, & E<E_{\text{th}}.\end{cases} \tag{34}$$

In this expression we have set

$$(i\lambda\!\!\!^{-}/2)\{[\sum_{l\neq l'}(2l+1)(1-\bar{U}_l)P_l]+(2l'+1)P_{l'}\}\equiv Fe^{i\beta},$$
$$\langle\sin 2\Delta\delta_l\rangle_{\text{av}}=0 \quad (\text{all } l),$$

and assumed no correlations between $\Delta\delta_l(l\neq l')$ and $\Delta\delta_{l'}$. Also we have kept in mind the remarks made in footnote (16) in placing the averaging bars over $\sigma_r^{(l')}$. If $\delta_{l'}$ has reasonable small rms fluctuations we can set $\cos 2\Delta\delta_{l'}\cong 1$. This would indicate that below-threshold effects should be noticeable in many situations. Equation (34) reduces to the extreme cases (21) and (16) if either $\Delta\delta_l\equiv 0$ or $\bar{\delta}_l\equiv 0$ for all l. Expressions for the integrated cross sections are obtained by making the substitution (33) in Eqs. (22) and (23) and taking energy averages.

One question of interest is the energy range over which expressions like (16) and (21) are valid. It appears from the derivation of Eq. (16) that there is no fundamental limitation of its range of validity. On the other hand, Eq. (21) is always derived[11] under the assumption that $k_0a\ll 1$ (a= channel radius). We have shown, admittedly not rigorously, that Eq. (16) can be obtained by energy averaging Eq. (21). This might indicate that expressions like (21) are valid over a larger energy range than is implied by $k_0a\ll 1$. Experimentally[21,22,23] Wigner cusps have been found to extend over energy regions of approximately 100 keV from threshold, but no detailed theoretical fits appear to have been made, except in one case.[23]

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[21] P. R. Malmberg, Phys. Rev. **101**, 114 (1955).
[22] N. Jarmie and R. C. Allen, Phys. Rev. **114**, 176 (1959).
[23] H. W. Newson, R. M. Williamson, K. W. Jones, J. H. Gibbons, and H. Marshak, Phys. Rev. **108**, 1294 (1957).