Interaction of Photons and Gravitons*

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The Hamiltonian formulation of general relativity is employed to study the interaction of photons and gravitons in the first approximation. The redundant variables are eliminated by an appropriate choice of gauge and coordinate conditions. S-matrix elements are calculated for initial states in which one photon is present and final states in which a photon and a graviton are present. Self-energy effects appear in first order but contribute nothing. Energy and momentum can be strictly conserved only if the initial and final photons and the gravitons all propagate in the same direction. For this case the S-matrix elements vanish in consequence of the null character of the Maxwell field, and the transition probability is also zero. Energy need not be exactly conserved if the process occurs at a rate which is sufficiently high. Under these conditions, corresponding to energies$\gg 10^{28}$ eV, a photon might decay into another photon and a graviton. The graviton has very low energy. This cannot explain the red shift as a "tired light" phenomenon. The creation of gravitons by Coulomb scattering of photons and by scattering in a magnetostatic field is shown to occur and the cross sections are calculated.

INTRODUCTION

IN recent years a number of authors[1–4] have discussed the Hamiltonian formulation of general relativity. Quantization of the exact theory may lead to some important consequences. This is a difficult program. While it is being carried out, we attempt to extract new physics out of an approximate treatment using existing mathematical procedures. Here the interaction of gravitons and photons is studied in the first approximation. This was motivated in part by the fact that light reaches us from the most distant galaxies after a time $\sim 10^{17}$ sec. During this long interval it interacts with the other vacuum fields, including the gravitational field. Even a very weak interaction might lead to observable effects over such an incredibly long time.

Hamiltonian Formulation of Gravitation and Electromagnetism

We start with the Hamiltonian formulation of Einstein's 1916 theory, in the canonical form[3] with the Hamiltonian given by

$$H=\int[(-g^{00})^{-1/2}\mathcal{H}_L+g_{r0}e^{rs}\mathcal{H}_s]d^3x. \tag{1}$$

Repeated Greek indices are summed over 0, 1, 2, 3; repeated Latin indices are summed over 1, 2, 3, except for the index k which usually indicates a sum over propagation modes. In (1), $g_{\mu\nu}$ is the metric tensor. H_L and H_s have both gravitational and nongravitational parts. e^{rs} is the inverse of the spatial metric g_{rs}.

We assume weak fields and write

$$g_{\mu\nu}=\delta_{\mu\nu}+h_{\mu\nu}. \tag{2}$$

Here $\delta_{\mu\nu}$ is the Lorentz metric and $h_{\mu\nu}$ is a first-order quantity. $\pi'^{\mu\nu}$ will designate the momenta canonically conjugate to $g_{\mu\nu}$. The $\pi'^{\mu0}$ vanish in consequence of the constraints. Coordinates may be chosen such that[5]

$$g_{\mu0}=\delta_{\mu0}. \tag{3}$$

This leaves us then with the six gravitational field variables g_{rs} and the momenta π'^{rs}. A Fourier decomposition is now carried out, assuming the fields are in a box of volume V.

$$g_{rs}=V^{-1/2}h_{krs}\exp(ik_mx^m)+\delta_{rs}, \tag{4}$$

$$\pi'^{rs}=V^{-1/2}\pi_k'^{rs}\exp(-ik_mx^m). \tag{5}$$

In (4) and (5) a summation over all propagation modes k is implied. We omit such summation signs, since it is usually evident when a mode sum is required.

The three-vector k_m defines a space direction for an allowed mode of propagation. Let U_{k1} and U_{k2} be two unit vectors orthogonal to each other and to k_m. We write h_{k11}, h_{k12}, h_{k22}, for the components of h_{krs} required by the two degrees of freedom for each k. Note that the numerical subscripts refer to a reference triad associated with the given k_m.

The Hamiltonian for the gravitational field alone may be written in the weak-field approximation, in appropriate units, as

$$\begin{aligned}H_G=\pi_k'^{12}\pi_{-k}'^{12}+\pi_k'^{21}\pi_{-k}'^{21}+k^2h_{k12}h_{-k12}/4\\+k^2h_{k21}h_{-k21}/4-k^2h_{k11}h_{-k22}/4-k^2h_{k22}h_{-k11}/4\\+(\pi_k'^{11}-\pi_k'^{22})(\pi_{-k}'^{11}-\pi_{-k}'^{22})/2.\end{aligned} \tag{6}$$

In obtaining (6) we employed those constraints which require vanishing of field variables and momenta with coordinate subscripts in the direction of k_m (longitudinal).

The symmetry of the metric tensor requires

$$\pi'^{12}=\pi'^{21};\quad h_{k12}=h_{k21}. \tag{7}$$

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[1] F. A. E. Pirani and A. Schild, Phys. Rev. **79**, 986 (1950).

[2] P. G. Bergmann, R. Penfield, R. Schiller, and H. Zatzkis, Phys. Rev. **80**, 81 (1950).

[3] P. A. M. Dirac, Proc. Roy. Soc. (London) **246**, 333 (1958).

[4] R. Arnowitt, S. Deser, and C. W. Misner, Nuovo cimento **15**, 487 (1960).

[5] See, for example, C. Møller, *The Theory of Relativity* (Oxford University Press, New York, 1952), p. 296.

The additional constraint relations,

$$h_{k11}=-h_{k22}, \tag{8}$$

must be taken into account. If we work with (6), retaining h_{k11} and h_{k22} as separate variables, the constraint relations (8) also appear as consequences of the equations of motion. The real degrees of freedom are associated with h_{1k2} and $(h_{k11}-h_{k22})$. We may employ these as dynamical variables by carrying out the transformations

$$\pi^{12}=2\pi'^{12}, \tag{9}$$

$$q_k=h_{k11}-h_{k22}, \tag{10}$$

$$2\rho_k=\pi_\kappa'^{11}-\pi_k'^{22}. \tag{11}$$

The substitutions (9), (10), and (11) are essential to guarantee the correct time dependence for the new variables. The new Hamiltonian for the gravitational field alone is

$$H_G=\pi_k^{12}\pi_{-k}^{12}/2+k^2h_{k12}h_{-k12}/2+2\rho_k\rho_{-k}+k^2q_kq_{-k}/8, \tag{12}$$

with

$$\pi_k^{12}=\pi_{-k}^{*12},\quad h_{k12}=h_{-k}{}^*{}_{12},\quad \rho_k=\rho_{-k}{}^*,\quad q_k=q_{-k}{}^*.$$

The procedure which led to (12) is not entirely rigorous. This must not obscure the fact that this Hamiltonian correctly describes a system of gravitational plane waves, with the dynamical variables h_{k12}, $(h_{k11}-h_{k22})$, associated with the two degrees of freedom for each allowed k. This is identical with the description which results from starting with Einstein's field equations to develop the theory of plane gravitational waves in the weak field approximation. Indeed, we might have started with the requirements[6] imposed by $R_{\mu\nu}=0$, then (12) would follow.

For the Maxwell field the Lagrangian density is

$$\mathfrak{L}=-g^{\alpha\nu}g^{\beta\mu}(A_{\nu,\mu}-A_{\mu,\nu})(A_{\alpha,\beta}-A_{\beta,\alpha})(-g)^{1/2}=-(-g)^{1/2}F_{\mu\nu}F^{\mu\nu}, \tag{13}$$

where g is the determinant of the metric tensor, A_μ is the four-potential, and $F_{\mu\nu}$ is the Maxwell field tensor. The momenta of the electromagnetic field are given by $\eta^\mu=\partial\mathfrak{L}/\partial A_{\mu,0}$, with

$$\eta^\alpha=-4g^{\alpha\mu}g^{0\nu}F_{\nu\mu}(-g)^{1/2}=-4(-g)^{1/2}F^{0\alpha}. \tag{14}$$

The component η^0 vanishes identically.

The Maxwell-field Hamiltonian density is then

$$\mathfrak{H}_M=\eta^dA_{d,0}+g^{\alpha\mu}g^{\beta\nu}F_{\beta\alpha}F_{\nu\mu}(-g)^{1/2}. \tag{15}$$

To reduce (15) further, we can proceed as follows:

$$\int\eta^dA_{0,d}d^3x=\int[(\eta^dA_0)_{,d}-\eta^d{}_{,d}A_0]d^3x. \tag{16}$$

The first term on the right of (16) vanishes because it may be written as a surface integral. The second term on the right is omitted because of the Maxwell equations $\eta^l{}_{,l}=0$. Expression (16) then becomes

$$\int\eta^lA_{0,l}d^3x=0. \tag{17}$$

The components F_{0l} may be written

$$F_{0l}=-\frac{g_{ml}\eta^m}{4g^{00}(-g)^{\frac{1}{2}}}-\frac{g_{ml}g^{mr}g^{0n}F_{nr}}{g^{00}}. \tag{18}$$

Employing (14), (17), and (18) reduces (15) to the form

$$\mathfrak{H}_M=(-g^{00})^{-\frac{1}{2}}\left[\frac{g_{lr}\eta^l\eta^r}{8(^3g)^{1/2}}+(^3g)^{1/2}e^{rs}e^{lm}F_{lr}F_{ms}\right]+e^{ls}g_{s0}F_{lm}\eta^m, \tag{19}$$

where (^3g) is the determinant of the spatial metric g_{ij}. Equation (19) is in the canonical form given by Dirac.[3] The variable A_0 no longer appears in the Hamiltonian. $F_{ij}=A_{j,i}-A_{i,j}$. $F_{i0}=-A_{i,0}$.

We now employ (2) and (3) to write (19) in the approximate form,

$$\mathfrak{H}_M=\int[\eta^l\eta^l/8+F_{lm}F_{lm}]d^3x+\int[(h_{11}+h_{22}+h_{33})(F_{lm}F_{lm}/2-\eta^l\eta^l/16)+h_{lm}\eta^l\eta^m/8-2h_{df}F_{md}F_{lf}\delta^{ml}]d^3x. \tag{20}$$

The second integral of (20) represents the first-order interaction between the Maxwell field and the gravitational field. We now assume that the background geometry is a flat space with a Lorentz metric. The $h_{\mu\nu}$ then represent a field which is treated by standard procedures.

The free-field equations for A_i are obtained from (20) as

$$4(A_{j,ii}-A_{i,ji})=\dot{\eta}^j, \tag{21}$$

$$\eta^j=4\dot{A}_j. \tag{22}$$

We select the Coulomb gauge,[7]

$$A_{i,i}=0; \tag{23}$$

the A_i then satisfy

$$\Box A_i=0. \tag{24}$$

The procedure we have followed has resulted in elimination of the redundant variables in gravitation and electromagnetism. Its principal advantage is simplicity. This is obtained at the expense of obscuring the gauge invariance.

A Fourier decomposition of the Maxwell field is

[6] J. Weber, *General Relativity and Gravitational Waves* (Interscience Publishers, Inc., New York, 1961), p. 92.

[7] W. Heitler, *The Quantum Theory of Radiation* (Oxford University Press, New York, 1954), 3rd ed., Appendix 2.

carried out by writing

$$\eta_r = V^{-1/2}\eta_{kr}\exp(-ik_m x^m) = V^{-1/2}(\eta_{k1}U_{k1r}+\eta_{k2}U_{k2r})\exp(-ik_m x^m), \quad (25)$$

$$A_s = V^{-1/2}A_{ks}\exp(ik_m x^m) = V^{-1/2}(a_{k1}U_{k1s}+a_{k2}U_{k2s})\exp(ik_m x^m), \quad (26)$$

$$F_{ls} = V^{-1/2}i[A_{ks}k_l - A_{kl}k_s]\exp(ik_m x^m). \quad (27)$$

Here the object A_{kl} is the lth component of the Fourier component A_k.

The gauge condition (23) is satisfied in consequence of

$$U_{k1i}k_i = 0, \quad U_{k2i}k_i = 0.$$

Making use of (8), (12), and (25)–(27) enables us to write for the total Hamiltonian

$$H = \pi_k{}^{12}\pi_{-k}{}^{12}/2 + k^2 h_{k12}h_{-k12}/2 + 2\rho_k\rho_{-k} + k^2 q_k q_{-k}/8 + \eta_{k1}\eta_{-k1}/8 + \eta_{k2}\eta_{-k2}/8 + 2k^2(a_{k1}a_{-k1}+a_{k2}a_{-k2}) + \int (h_{rs}\eta_r\eta_s/8 - 2h_{lj}F_{ml}F_{dj}\delta^{md})d^3x. \quad (28)$$

Equation (28) is a Hamiltonian which gives, in first order, the correct equations of motion for the required dynamical variables, for the chosen coordinate system and gauge condition. It is seen to be that of sets of gravitational and electromagnetic field oscillators, together with coupling terms. For convenience we have not Fourier-analyzed the interaction part.

The a_k and h_{k12} become operators when the theory is quantized. In the Schrödinger picture, in a representation in which energy is diagonal, these have matrix elements such as $(a_k)_{mn}$ which vanish unless $m=n+1$ or $m=n-1$. We define new operators by the relations[8]

$$h_{k12} = (2k)^{-1/2}(b_{-k}{}^* + b_k), \quad (29)$$

$$\pi_{-k12} = i(k/2)^{1/2}(b_{-k}{}^* - b_k), \quad (30)$$

$$q_k = (2/k)^{1/2}(d_{-k}{}^* + d_k). \quad (31)$$

$$\rho_{-k} = i(k/8)^{1/2}(d_{-k}{}^* - d_k), \quad (32)$$

$$a_{k1} = (8k)^{-1/2}(f_{-k1}{}^* + f_{k1}), \quad (33)$$

$$\eta_{-k1} = i(2k)^{1/2}(f_{-k1}{}^* - f_{k1}), \quad (34)$$

$$a_{k2} = (8k)^{-1/2}(f_{-k2}{}^* + f_{k2}), \quad (35)$$

$$\eta_{-k2} = i(2k)^{1/2}(f_{-k2}{}^* - f_{k2}). \quad (36)$$

These operators have matrix elements $b_{n,n-1}=0$, $b_{n-1,n}{}^*=0$, $b_{n-1,n}=n^{1/2}$, $b_{n,n-1}{}^*=n^{1/2}$. b^* is therefore a graviton creation operator, and b is an annihilation operator for the k_{12} type of graviton. Similarly d^* creates gravitons of the second independent state of polarization, d annihilates them. f^* and f are photon creation and annihilation operators, respectively. Making use of the commutation rules $[h_{k12},\pi_{k'}{}^{12}] = i\delta_{kk'}$ and dropping the zero-point energy enables us to write the Hamiltonian as

$$H = k(b_k{}^*b_k + d_k{}^*d_k + f_{k1}{}^*f_{k1} + f_{k2}{}^*f_{k2}) + \int [(h_{rs}\eta^r\eta^s/8) - 2h_{lj}F_{ml}F_{dj}\delta^{md}]d^3x. \quad (37)$$

As we remarked earlier, all of the operators appearing in (37) are Schrödinger operators. The interaction picture (representation) operators are obtained in the usual way by writing, for example,

$$b_{k\mathrm{IR}} = b_k e^{-ikt}, \quad b_{k\mathrm{IR}}{}^* = b_k{}^* e^{ikt}. \quad (38)$$

Operators of the interaction picture will be employed in what follows and the subscript IR omitted.

The first order S-matrix elements may now be written for a process in which there is initially one photon present and no gravitons, to a final state in which one photon and one graviton are present as

$$M_{BC,A} = -i\int \langle 0|(\beta_1 f_{k1B}+\beta_2 f_{k2B}) \times (\gamma_1 b_{kC}+\gamma_2 d_{kC})|(h_{ij}\eta^i\eta^j/8) - 2h_{ij}F_{lj}F_{mi}\delta^{ml}|(\alpha_1 f_{k1A}{}^*+\alpha_2 f_{k2A}{}^*)|0\rangle d^4x. \quad (39)$$

In (39), A labels the initial photon, f_{k1B} is an annihilation operator for photons propagating in the k direction and polarized in the U_{k1} direction. B is a label for the final state. Similarly b_{kC} and d_{kC} are annihilation operators for the two kinds of gravitons. C is a label for the final-state graviton. α_1, α_2, β_1, β_2, γ_1, γ_2 are constants needed to describe mixed states of polarization. Equation (39) is in fact gauge invariant since η^i is the electric field and F_{lj} is the magnetic field.

To evaluate (39) we note that for each direction defined by k there are four field variables corresponding to two states of polarization for the gravitational field and the electromagnetic field. h_{ij} transforms like a tensor under Lorentz transformations. Consider a Lorentz frame with coordinates x^0, y^1, y^2, y^3. y^3 is measured in a direction parallel to a given propagation three-vector $\mathbf{k}$. y^1 and y^2 are measured in the two space directions orthogonal to each other and to $\mathbf{k}$. h_{k11} is the 11 component of h_k in the y frame. To obtain h_{ij} in the observer's frame with observer's coordinates x^0, x^1, x^2, x^3, we recall that

$$h_{kij} = h_{k11}\frac{\partial y_k{}^1}{\partial x^i}\frac{\partial y_k{}^1}{\partial x^j} + h_{k22}\frac{\partial y_k{}^2}{\partial x^i}\frac{\partial y_k{}^2}{\partial x^j} + h_{k12}\left[\frac{\partial y_k{}^1}{\partial x^i}\frac{\partial y_k{}^2}{\partial x^j} + \frac{\partial y_k{}^2}{\partial x^i}\frac{\partial y_k{}^1}{\partial x^j}\right], \quad (40)$$

not summed over k.

[8] We have omitted absolute value signs on k here and in what follows. Whenever k appears without a lower case subscript, it is either a label or the absolute value of the three-vector k_i is implied. k with an upper case subscript labels a particular particle. An asterisk denotes the adjoint operator.

Making use of (8) as an operator identity gives

$$h_{kij}=\frac{q_k}{2}\left[\frac{\partial y_k^1}{\partial x^i}\frac{\partial y_k^1}{\partial x^j}-\frac{\partial y_k^2}{\partial x^i}\frac{\partial y_k^2}{\partial x^j}\right]+h_{k12}\left[\frac{\partial y_k^1}{\partial x^i}\frac{\partial y_k^2}{\partial x^j}+\frac{\partial y_k^2}{\partial x^i}\frac{\partial y_k^1}{\partial x^j}\right], \quad (41)$$

not summed over k.

We may express h_{kij} in terms of the unit vectors U_{k1} and U_{k2} which were defined earlier,

$$U_{k1i}=\partial y_k^1/\partial x^i, \quad U_{k2i}=\partial y_k^2/\partial x^i. \quad (42)$$

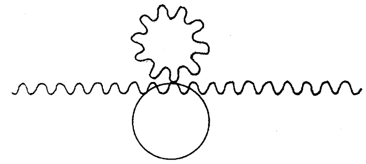

FIG. 1. Feynman diagram for virtual photon graviton process.

Employing (42), (29), and (31) in (41) gives

$$h_{kij}=(2k)^{-1/2}[(d_{-k}^*+d_k)(U_{k1i}U_{k1j}-U_{k2i}U_{k2j})+(b_{-k}^*+b_k)(U_{k1i}U_{k2j}+U_{k2i}U_{k1j})]. \quad (43)$$

The S matrix elements (39) then assume the form

$$\begin{aligned}M_{BC,A}=&-iV^{-3/2}\int\langle 0|(\beta_1 f_{k1B}+\beta_2 f_{k2B})(\gamma_1 b_{kC}+\gamma_2 d_{kC})|\{(2k')^{-1/2}[(d_{-k'}^*+d_{k'})(U_{k'1i}U_{k'1j}-U_{k'2i}U_{k'2j})\\&+(b_{-k'}^*+b_{k'})(U_{k'1i}U_{k'2j}+U_{k'2i}U_{k'1j})]\}\{-(k''k'''/16)^{1/2}[(f_{-k''1}-f_{k''1})U_{k''1i}+(f_{-k''2}^*-f_{k2''})U_{k''2i}]\\&\times[(f_{-k'''1}^*-f_{k'''1})U_{k'''1j}+(f_{-k'''2}^*-f_{k'''2})U_{k'''2j}]+(16k''k''')^{-1/2}[(f_{-k''1}^*+f_{k''1})(k''_l U_{k''1j}-k_j''U_{k''1l})\\&+(f_{-k''2}^*+f_{k''2})(k_l''U_{k''2j}-k_j''U_{k''2l})][(f_{-k'''1}^*+f_{k'''1})(k_l'''U_{k'''1i}-k_i'''U_{k'''1l})\\&+(f_{-k'''2}^*+f_{k'''2})(k_l'''U_{k'''2i}-k_i'''U_{k'''2l})]\}\exp[ix^\mu(k_\mu'+k_\mu''+k_\mu''')]|(\alpha_1 f_{k1A}^*+\alpha_2 f_{k2A}^*)|0\rangle d^4x, \quad (44)\end{aligned}$$

summed over k', k'', and k'''. Here k_μ is the propagation four-vector. Carrying out some of these operations leads to

$$\begin{aligned}M_{BC,A}=&-i(2\pi)^4V^{-3/2}\delta_4(k_\mu'+k_\mu''+k_\mu''')(\beta_1 f_{k1B}+\beta_2 f_{k2B})\{[\gamma_1\delta_{kC,-k'}(U_{k'1i}U_{k'2j}+U_{k'2i}U_{k'1j})\\&+\gamma_2\delta_{kC,-k'}(U_{k'1i}U_{k'1j}-U_{k'2i}U_{k'2j})](2k')^{-1/2}\}\{-(k''k'''/16)^{1/2}[(f_{-k''1}^*-f_{k''1})U_{k''1i}\\&+(f_{-k''2}^*-f_{k''2})U_{k''2i}][(f_{-k'''1}^*-f_{k'''1})U_{k'''1j}+(f_{-k'''2}^*-f_{k'''2})U_{k'''2j}]\\&+(16k''k''')^{-1/2}[(f_{-k''1}^*+f_{k''1})(k_l''U_{k''1j}-k''_jU_{k''1l})+(f_{-k''2}^*+f_{k''2})(k_l''U_{k''2j}-k_j''U_{k''2l})]\\&\times[(f_{-k'''1}^*+f_{k'''1})(k_l'''U_{k'''1i}-k_i'''U_{k'''1l})+(f_{-k'''2}^*+f_{k'''2})(k_l'''U_{k'''2i}-k_i'''U_{k'''2l})]\}\\&\times(\alpha_1 f_{k1A}^*+\alpha_2 f_{k2A}^*). \quad (45)\end{aligned}$$

We return for a moment to expression (44) and note that it involves some products of the form

$$\langle 0|f_{kB}|d_{kC}d_{k'}^*f_{k''}f_{k'''}^*|f_{kA}^*|0\rangle. \quad (46)$$

In one possible sequence $f_{k''}$ annihilates $f_{k'''}^*$, f_{kB} annihilates f_{kA}^*, $d_{k'}^*$ creates a graviton and d_{kC} annihilates it. Initially we have a photon and in the final state the same photon. The Feynman (self-energy) diagram is given by Fig. 1. Let us evaluate (45) for this kind of process by writing first

$$\begin{aligned}M_{BC,A}=&-i(2\pi)^4V^{-3/2}\delta_4(k_\mu'+k_\mu''+k_\mu''')[\beta_1 f_{k1B}+\beta_2 f_{k2B}]\{[\delta_{kC,-k'}\gamma_1(U_{k'1i}U_{k'2j}+U_{k'2i}U_{k'1j})\\&+\gamma_2\delta_{kC,-k'}(U_{k'1i}U_{k'1j}-U_{k'2i}U_{k'2j})](2k')^{-1/2}\}[U_{k''1i}U_{-k''1j}k''+U_{k''2i}U_{-k''2j}k''\\&+k''^{-1}(k_l''U_{k''1j}-k_j''U_{k''1l})(k_i''U_{-k''1l}-k_l''U_{-k''1i})+k''^{-1}(k_l''U_{k''2j}-k_j''U_{k''2l})\\&\times(k_i''U_{-k''2l}-k_l''U_{-k''2i})](\alpha_1 f_{k1A}^*+\alpha_2 f_{k2A}^*)/4. \quad (47)\end{aligned}$$

Carrying out these operations gives

$$\begin{aligned}M_{BC,A}=&-i(2\pi)^4V^{-3/2}\delta_4(k_C)[\beta_1 f_{k1B}+\beta_2 f_{k2B}]\{[\gamma_1(U_{-kC1i}U_{-kC2j}+U_{-kC2i}U_{-kC1j})\\&+\gamma_2(U_{-kC1i}U_{-kC1j}-U_{-kC2i}U_{-kC2j})](2k_C)^{-1/2}\}\{k''(U_{k''1i}U_{-k''1j}+U_{k''2i}U_{-k''2j})\\&+k''^{-1}[-k_i''k_j''(U_{k''1l}U_{-k''1l}+U_{k''2l}U_{-k''2l})\\&-k''^2(U_{-k''2i}U_{k''2j}+U_{-k''1i}U_{k''1j})]\}(\alpha_1 f_{k1A}^*+\alpha_2 f_{k2A}^*). \quad (48)\end{aligned}$$

(48) vanishes in consequence of

$$U_{k''1l}U_{-k''1l}+U_{k''2l}U_{-k''2l}=0,$$

except perhaps at $k_C=0$ where (48) is not defined. However, the square of (48) will in any case vanish when integrated over momentum space because of the part of the phase space volume element $k_C^2dk_CV$, if the volume is allowed to become infinite. For a finite volume, at small k_C the appropriate phase volume element is Ldk_C, where L is the length of the box; but k_C cannot be zero in this case so the phase volume integral of the square of (48) is zero.

We turn to the other kinds of terms in (45). Consider the one polarization first, for the group of terms:

$$\beta_1 f_{k1B}(16k''k''')^{-1/2}(f_{-k''1}{}^*+f_{k''1})(k_l''U_{k''1j}-k_j''U_{k''1l})(f_{-k'''1}{}^*+f_{k'''1})(k_l'''U_{k'''1i}-k_i'''U_{k'''1l})\alpha_1 f_{k1A}{}^*$$
$$=\alpha_1\beta_1(16k''k''')^{-1/2}(k_l''U_{k''j}-k_j''U_{k''l})(k_l'''U_{k'''i}-k_i'''U_{k'''l})[f_{k1B}(f_{-k''}{}^*f_{k'''}+f_{k''}f_{-k'''}{}^*)f_{k1A}{}^*].$$

Now

$$f_{k1B}(f_{k''}f_{-k'''}{}^*+f_{-k''}{}^*f_{k'''})f_{k1A}{}^*=\delta_{k1B,-k''}\delta_{k''',k1A}+\delta_{k1B,-k'''}\delta_{k1A,k''}.$$

Employing this and carrying out the sum over k'' and k''' then gives for these terms

$$(\alpha_1\beta_1/4)[(k_Ak_B)^{-1/2}(k_{-Bl}U_{-kBj}-k_{-Bj}U_{-k1Bl})(k_{Al}U_{kA1i}-k_{Ai}U_{kA1l})$$
$$+(k_Bk_A)^{-1/2}(k_{Al}U_{kA1j}-k_{Aj}U_{kA1l})(k_{-Bl}U_{-kB1i}-k_{-Bi}U_{-kB1l})]. \quad (49)$$

A similar contribution results for the other polarization. We then evaluate the terms:

$$(-\alpha_1\beta_1/4)(k''k''')^{1/2}U_{k''1i}U_{k'''1j}f_{k1B}[f_{-k''1}{}^*-f_{k''1}][f_{-k'''1}{}^*-f_{k'''1}]f_{k1A}{}^*$$
$$=(\alpha_1\beta_1/4)(k_Bk_A)^{1/2}[U_{k1Ai}U_{-kB1j}+U_{-kB1i}U_{kA1j}]. \quad (50)$$

A similar contribution results from the other polarization. Also there will be some terms resulting from annihilation of a photon of one polarization and creation of a photon of the other polarization. Assembling these results gives us for the S matrix elements

$$M_{BC,A}=-i(2\pi)^4V^{-3/2}\delta_4(k_{A\mu}-k_{B\mu}-k_{C\mu})(2k_C)^{-1/2}(2\gamma_1U_{-kC1i}U_{-kC2j}+\gamma_2[U_{-kC1i}U_{-kC1j}-U_{-kC2i}U_{-kC2j}])$$
$$\times[(\alpha_1\beta_1/4)\chi_{11}+(\alpha_2\beta_1/4)\chi_{21}+(\alpha_1\beta_2/4)\chi_{12}+(\alpha_2\beta_2/4)\chi_{22}],$$

with

$$\chi_{\Gamma\Omega}=(k_Ak_B)^{1/2}[(U_{k\Gamma Ai}U_{-k\Omega Bj}+U_{-k\Omega Bi}U_{k\Gamma Aj})]+(k_Ak_B)^{-1/2}[(k_{-Bl}U_{-k\Omega Bj}-k_{-Bj}U_{-k\Omega Bl})(k_{Al}U_{k\Gamma Ai}-k_{Ai}U_{k\Gamma Al})$$
$$+(k_{Al}U_{k\Gamma Aj}-k_{Aj}U_{k\Gamma Al})(k_{-Bl}U_{-k\Omega Bi}-k_{-Bi}U_{-k\Omega Bl})]. \quad (51)$$

The indices Γ, Ω run over 1 and 2 only.

In order to calculate the transition probability we must square (51) and integrate over momentum space. The occurrence of the delta function means, among other things, that the three (three-space) momentum vectors k_{Ai}, k_{Bi}, and k_{Ci} all lie in a plane, and sum to zero. It is useful to evaluate (51) for this condition, shown in Fig. 2. U_1 is normal to the plane of the figure, U_2 lies in the plane. In this case,

$$k_{-Bj}k_{Ai}U_{-kC2i}U_{-kC2j}=-\sin\varphi\sin(\theta+\varphi). \quad (52)$$

Evaluating (51) for both polarizations by use of (52) leads to

$$M_{BC,A}=-i(2\pi)^4V^{-3/2}\delta_4(k_{A\mu}-k_{B\mu}-k_{C\mu})(k_Ak_B/8k_C)^{1/2}\{\alpha_1\beta_1\gamma_2[\cos\theta-1-\sin(\theta+\varphi)\sin\varphi]$$
$$+\alpha_2\beta_2\gamma_2[1-\cos\phi\cos(\theta+\varphi)]+\gamma_1(\alpha_1\beta_2+\alpha_2\beta_1)[\cos\varphi-\cos(\theta+\varphi)]\}, \quad (53)$$

with $k_{Ai}=k_{Bi}+k_{Ci}$.

Following the procedure of Lippmann and Schwinger[9] we square (51), set one of the delta functions equal to the quantity $(2\pi)^{-4}Vt$, and integrate over momentum space to obtain

$$W=(2\pi)^{-2}\int\delta_4(k_{A\mu}-k_{B\mu}-k_{C\mu})[\Phi_{11}+\Phi_{12}+\Phi_{22}]^2d^3k_Bd^3k_C. \quad (54)$$

Equation (54) is the transition probability for a photon to decay into a photon and a graviton. Φ_{11}, Φ_{12}, and Φ_{22} are parts of (51) contributed by the two possible polarizations of the initial photon. Evaluating (54) gives $k_{Ai}=k_{Bi}+k_{Ci}$ and making use of (53) then leads to

$$W=(2\pi)^{-2}\int\delta_0(k_A-k_B-k_C)(k_Ak_B/8k_C)\{\alpha_1\beta_1\gamma_2[\cos\theta-1-\sin(\theta+\varphi)\sin\varphi]$$
$$+\alpha_2\beta_2\gamma_2[1-\cos\varphi\cos(\theta+\varphi)]+\gamma_1(\alpha_1\beta_2+\alpha_2\beta_1)[\cos\varphi-\cos(\theta+\varphi)]\}^2d^3k_C, \quad (55)$$

[9] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

with

$$k_C^2=k_A^2+k_B^2-2k_Ak_B\cos\theta, \tag{56}$$

$$k_B^2=k_A^2+k_C^2-2k_Ak_C\cos\varphi. \tag{57}$$

Let

$$k=k_A-k_B-k_C. \tag{58}$$

Combining (56), (57), and (58) gives

$$\begin{aligned} k&=k_A-k_B-(k_A^2+k_B^2-2k_Ak_B\cos\theta)^{1/2},\\ k&=k_A-k_C-(k_A^2+k_C^2-2k_Ak_C\cos\varphi)^{1/2}. \end{aligned} \tag{59}$$

Noting that $d^3k_C=2\pi k_C^2dk_C\sin\varphi d\varphi$ enables us to write (55) in the form

$$\begin{aligned} W=(2\pi)^{-1}\int \delta_0(k)(k_Ak_Bk_C/8)\\ \times\{\alpha_1\beta_1\gamma_2[\cos\theta-1-\sin(\theta+\varphi)\sin\varphi]\\ +\alpha_2\beta_2\gamma_2[1-\cos\varphi\cos(\theta+\varphi)]\\ +\gamma_1(\alpha_1\beta_2+\alpha_2\beta_1)[\cos\varphi-\cos(\theta+\varphi)]\}^2\\ \times\left(\frac{\partial k_C}{\partial k}\right)_{\varphi=\text{constant}} dk\sin\varphi d\varphi. \end{aligned} \tag{60}$$

Employing (59) to evaluate $(\partial k_C/\partial k)_{\varphi=\text{constant}}$ gives

$$\partial k_C/\partial k=(k_C+k-k_A)/[k_A(1-\cos\varphi)-k]. \tag{61}$$

In order to evaluate the integral (60) it is helpful to consider several cases.

(a) Neither k_B nor k_C approach zero. Integrating with respect to k gives

$$\begin{aligned} W=(2\pi)^{-1}\int(k_C-k_A)[8k_A(1-\cos\varphi)]^{-1}k_Ak_Bk_C\\ \times\{\gamma_2\alpha_1\beta_1[\cos\theta-1-\sin(\theta+\varphi)\sin\varphi]\\ +\alpha_2\beta_2\gamma_2[1-\cos\varphi\cos(\theta+\varphi)]+\gamma_1(\alpha_1\beta_2+\alpha_2\beta_1)\\ \times[\cos\varphi-\cos(\theta+\varphi)]\}^2\sin\varphi d\varphi. \end{aligned} \tag{62}$$

From Fig. 2 we have $\sin\varphi=(k_B/k_C)\sin\theta$. (62) must be evaluated at $k=0$. From (59) and (58) this requires $\varphi=0$, $\theta=0$, corresponding to strict conservation of both momentum and energy. The integral is seen to vanish. The vanishing of the S-matrix elements for forward scattering is readily understood if we study the structure of the interaction for this case. It is made up of terms such as

$$h_{11}(E_1'E_1''-H_2'H_2''),$$

where E and H are the electric and magnetic field operators, respectively. For a null field $E^2-H^2=0$. On the other hand, if the scattered photon does not propagate in the same direction as the incident one, $E_1''\neq H_2''$ and the cancellation need not occur.

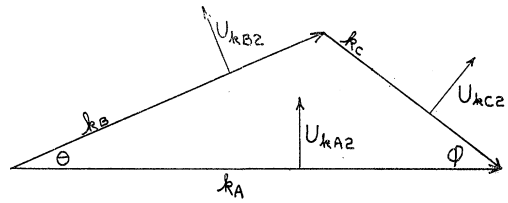

FIG. 2. Momenta and reference triads. U_{kA1}, U_{kB1}, U_{kC1} are normal to the figure.

(b) $k_C\to 0$, $k_B\neq 0$, φ need not vanish. Under circumstances for which (62) is valid a vanishing result would still be obtained as $k_C\to 0$. However, (62) is valid only for long times t such that

$$kt>\pi. \tag{63}$$

The origin of the delta function in (60) is known from time-dependent perturbation theory to be

$$\lim_{t\to\infty}\frac{2\sin^2(kt/2)}{\pi k^2t}\to\delta(k). \tag{64}$$

For

$$kt\ll\pi \tag{65}$$

the delta function can be replaced by $t/2\pi$. For this case we may approximately evaluate (60) as

$$\begin{aligned} W=(k_A^2t/32\pi^2)\int_0^k\int_0^\pi[\gamma_2(\alpha_1\beta_1-\alpha_2\beta_2)]^2k_C\\ \times\left(\frac{\partial k_C}{\partial k}\right)_{\varphi=\text{constant}}\sin^5\varphi d\varphi dk. \end{aligned} \tag{66}$$

From (59), for small k_C,

$$k\approx k_C(\cos\varphi-1), \tag{67}$$

the integral (66) is evaluated as

$$W=tk_A^2k^2[\gamma_2(\alpha_1\beta_1-\alpha_2\beta_2)]^2/24\pi^2\approx t^{-1}. \tag{68}$$

We now insert dimensional factors and, from the inequality (65), (68) becomes

$$k_A\gg(c^3/\hbar G)^{1/2}. \tag{69}$$

(69) is the condition to be met for decay of a photon into another photon and a low-energy graviton. It requires energies $\gg 10^{28}$ electron volts.

(c) $k_B\to 0$, $k_C\neq 0$, $\varphi\to 0$, θ need not be zero. The integral (60) tends to

$$\begin{aligned} W\approx(tk^2k_A^2/64\pi^2)\int(1-\cos\theta)\\ \times(\gamma_2\alpha_2\beta_2-\gamma_2\alpha_1\beta_1+\gamma_1\alpha_1\beta_2+\gamma_1\alpha_2\beta_1)^2\sin\varphi d\varphi. \end{aligned} \tag{70}$$

In this order of approximation $W\to 0$ in consequence of the condition $\varphi\to 0$. We may therefore conclude that processes for which a photon decays into another

photon and a graviton are excluded in this approximation, except possibly at extreme energies, i.e., $\gg 10^{28}$ eV, corresponding to (69). In these cases the photon may conceivably decay into a very low energy graviton and a photon of almost the same energy as the incident photon. This cannot explain the red shift on the "tired light" hypothesis, since it does not occur at low and intermediate energy.

Scattering by the Longitudinal Fields of Charges and Currents

A longitudinal electromagnetic field would be expected to produce gravitons when photons are incident upon it.

We may write for an electric field

$$\eta_r = \eta_{rT} + \eta_{rL}. \quad (71)$$

In (71), η_{rT} is the transverse part of the canonical momentum and η_{rL} is the longitudinal part.

The reduction of (16) to (17) would not follow since $(-g)^{-1/2}\eta^i{}_{,i} = J_0$, the charge density. (20) and (21) would be modified and (22) would be replaced by

$$\eta^j = 4(\mathring{A}_j - A_{0,j}).$$

We may now write for the Fourier decomposition

$$\eta_{Lr} = V^{-1/2}\eta_{kLr}\exp(ik_m x^m) = V^{-1/2}\eta_{kL}U_{kLr}\exp(ik_m x^m). \quad (72)$$

U_{kLr} is a unit vector in the direction of η_{Lr}. These expressions are summed over (k).

The longitudinal field gives a contribution to the S matrix for initial states in which one photon is present and no gravitons to final states in which no photons and one graviton are present.

$$M_{C,A} = V^{-3/2}\int \langle 0|\gamma_1 b_{kC} + \gamma_2 d_{kC}|(2k')^{-1/2}[(d_{-k'}{}^* + d_{k'})(U_{k'1i}U_{k'1j} - U_{k'2i}U_{k'2j}) + 2(b_{-k'}{}^* + b_{k'})U_{k'1i}U_{k'2j}]$$
$$\times[(k''/32)^{1/2}(U_{k''1i}f_{k''1} + U_{k''2i}f_{k''2})U_{kLj}\eta_{kL} + (k''/32)^{1/2}(U_{k''1j}f_{k''1} + U_{k''2j}f_{k''2})U_{kLi}\eta_{kL}]$$
$$\times\exp[ix^\mu(k_\mu' + k_\mu'' + k_{L\mu})]|\alpha_1 f_{k1A}{}^* + \alpha_2 f_{k2A}{}^*|0\rangle d^4x. \quad (73)$$

Evaluating (73) gives

$$M_{C,A} = (2\pi)^4 V^{-3/2}\delta_4(k_{A\mu} + k_{L\mu} - k_{C\mu})\eta_{kL}F_E(\alpha_1,\alpha_2,\gamma_1,\gamma_2,U_{kA1},U_{kA2},U_{-kC1},U_{-kC2}), \quad (74)$$

with

$$F_E = [2\gamma_1 U_{-kC1i}U_{-kC2j} + \gamma_2(U_{-kC1i}U_{-kC1j} - U_{-kC2i}U_{-kC2j})]$$
$$\times[\alpha_1 U_{kA1i}U_{kLj} + \alpha_2 U_{k2Ai}U_{kLj} + \alpha_1 U_{kA1j}U_{kLi} + \alpha_2 U_{k2Aj}U_{kLi}](k_A/k_C)^{1/2}/8. \quad (75)$$

A sum over k_L is implied by (74). Replacing this sum by an integral and noting that k_L has a vanishing timelike component gives

$$M_{C,A} = 2\pi V^{-1/2}\int \delta_4(k_{A\mu} + k_{L\mu} - k_{C\mu})\eta_{kL}F_E d^3k_L = 2\pi V^{-1/2}\delta_0(k_A - k_C)F_E\eta_{kL}|_{kLi = kCi - kAi}. \quad (76)$$

The transition probability is then obtained by integrating the square of (76) over momentum space, remembering that one of the delta functions is replaced by $t/2\pi$. In terms of the spherical coordinate k_C-space angles θ' and φ', we have

$$W = (2\pi)^{-2}\int k_C{}^2\delta_0(k_A - k_C)(F_E{}^2\eta_{kL}{}^2)_{kLi = kCi - kAi}\sin\theta' dk_C d\theta' d\varphi'. \quad (77)$$

After inserting dimensional factors, we may write this in the form[10]

$$W = \pi 2^8 G c^{-3} k_A{}^2 \langle \eta_{kL}{}^2\rangle_{\text{av}} \quad (78)$$

with $\langle \eta_{kL}{}^2\rangle_{\text{av}}$ given by

$$\langle \eta_{kL}{}^2\rangle_{\text{av}} = \frac{1}{4\pi}\int (\eta_{kL}{}^2 F_E{}^2)_{kLi = kCi - kAi}\sin\theta' d\theta' d\varphi'. \quad (79)$$

The differential cross section is

$$dS_{\text{Coulomb}} = 64 G c^{-4} V k_A{}^2 (\eta_{kL}{}^2 F_E{}^2)_{kLi = kCi - kAi}\sin\theta' d\theta' d\varphi', \quad (80)$$

with V the normalization volume for photons. The scattering cross section is therefore

$$S_{\text{Coulomb}} = \pi 2^8 G c^{-4} k_A{}^2 \langle \eta_{kL}{}^2\rangle_{\text{av}} V. \quad (81)$$

[10] In these units, $\eta^i = F^{i0}/4\pi$.

We note the absence of Planck's constant in (80) and (81). This is to be expected, since we have the interaction of two Boson fields, which has a classical limit. The classical calculation has been carried out and agrees with (80) and (81).

The case of graviton production by scattering of photons in a magnetostatic field can be carried through in a similar way. Now we have

$$A_s = A_{Ms} + A_{Ts}.$$

Here A_{Ms} represents the prescribed part of A, with vanishing time derivative. For the Fourier decomposition, A_{Ms} is given by

$$A_{Ms} = V^{-1/2} a_{kM} U_{kMs} \exp(ik_m x^m).$$

Corresponding to (76), we now obtain

$$M_{C,A} = 2\pi V^{-1/2} \delta_0(k_A - k_C) F_M (k_A k_C)^{-1/2} a_{kM} |_{kMi = kCi - kAi}, \tag{76A}$$

where

$$F_M = [-\gamma_1(U_{-kC1i}U_{-kC2j} + U_{-kC1j}U_{-kC2i}) - \gamma_2(U_{-kC1i}U_{-kC1j} - U_{-kC2i}U_{-kC2j}][k_{Ml}U_{kMj} - k_{Mj}U_{kMl}]$$
$$\times[\alpha_1(k_{Al}U_{kA1i} - k_{Ai}U_{kA1l}) + \alpha_2(k_{Al}U_{kA2i} - k_{Ai}U_{kA2l})].$$

Corresponding to (78),

$$W = \pi^{-1} G c^{-3} \langle a_{kM}^2 \rangle_{\text{av}}, \tag{78A}$$

with $\langle a_{kM}^2 \rangle_{\text{av}}$ given by

$$\langle a_{kM}^2 \rangle_{\text{av}} = \frac{1}{4\pi} \int a_{kM}^2 F_M^2 |_{kMi = kCi - kAi} \sin\theta' d\theta' d\varphi'. \tag{79A}$$

The differential cross section is

$$dS_{\text{Magnetic field}} = G(2\pi)^{-2} c^{-4} V (a_{kM}^2 F^2)_{kMi = kCi - kAi} \sin\theta' d\theta' d\varphi'. \tag{80A}$$

The scattering cross section is

$$S = \pi^{-1} G c^{-4} V \langle a_{kM}^2 \rangle_{\text{av}}. \tag{81A}$$

η_{kL}^2 and a_{kM}^2 contain V^{-1} so these results are independent of V. A closer study of F^2 indicates that the total scattering cross section for a large volume containing a uniform field tends to

$$S \approx 8\pi^2 G U l c^{-4},$$

where U is the energy and l is the linear dimension of the scatterer.

For laboratory experiments the cross section appears too small. Thus, for a volume 10^6 cc containing 10^{15} ergs of electrical energy, we have

$$S \to 10^{-30} \text{ cm}^2.$$

On the other hand, for a galaxy containing a magnetic field $\sim 10^{-6}$ G, with $l \sim 10^{22}$ cm,

$$S \to 10^{28} \text{ cm}^2.$$

A fraction $10^{28}/10^{44} = 10^{-16}$ of the incident photons would be converted to gravitons.

CONCLUSIONS

We have studied the interaction of gravitons and photons in the first approximation. Some simplification results from use of a coordinate system and a gauge condition in which all of the redundant variables have been eliminated. In first order self-energy effects appear but these contribute nothing. In this order the decay of a photon into a low-energy graviton and a photon might occur at photon wavelengths

$$\lambda \ll (G\hbar/c^3)^{1/2},$$

corresponding to energies $\gg 10^{28}$ eV. This does not constitute a possible "tired light" mechanism.

Electrostatic and magnetostatic fields may annihilate a photon with production of a graviton.