it is plausible that some other types of excitation will then be located at fairly small fractions of m_0. Thus, one could anticipate that the known spin-0 bosons, for example, are secondary dynamical manifestations of strongly coupled primary fermion fields and vector gauge fields. This line of thought emphasizes that the question "Which particles are fundamental?" is incorrectly formulated. One should ask "What are the fundamental fields?"

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Scattering of Electromagnetic Waves in Saxon-Schiff Theory

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We calculate the diffraction of electromagnetic waves by weak scatterers with complex dielectric constant and permeability using the Saxon-Schiff theory of potential scattering. Boundary conditions, polarizations, and the optical theorem are discussed to some extent. Our results for the scattering amplitude contain certain special cases obtained previously by other authors. In an Appendix, we compare the results for the scattering by a homogeneous dielectric sphere with those of the exact Mie theory. It is seen that the Saxon-Schiff theory gives a good qualitative agreement insofar as it reproduces the diffraction maxima and minima, in vast superiority to the Born approximation. In the asymptotic limit $kR\rightarrow\infty$, the radar cross section is shown to agree with the exact result for a not too large index of refraction.

THE theory of Saxon and Schiff,[1] originally developed for high-energy scalar potential scattering, has been applied to the scattering of electromagnetic waves by dielectric bodies.[2] Schiff[3] has also considered scattering of vector waves using an earlier version of the theory, valid for either small or large angles only. In this note, we derive the scattering amplitude of electromagnetic waves for a general weak scatterer with complex dielectric constant and permeability, and demonstrate that the results can be made to reduce to the large- and small-angle expressions of Schiff[3] in the respective limits.

Maxwell's equations, setting $c=1$ and assuming a harmonic time dependence of the fields,

$$\sim\exp(-ikt),$$

become

$$\nabla\times\mathbf{E}=ik\mu\mathbf{H},\quad \nabla\times\mathbf{H}=(\sigma-ik\epsilon)\mathbf{E}. \tag{1}$$

No free charges are assumed to be present; σ is the conductivity, and ϵ, μ are dielectric constant and permeability, respectively (we shall use Gaussian units, $\epsilon_0=\mu_0=1$). Taking the divergence of the second equation, we get

$$\nabla\cdot\epsilon'\mathbf{E}=0, \tag{2}$$

where we have introduced the complex dielectric constant,

$$\epsilon'=\epsilon(1+i\nu),$$

with

$$\nu=\sigma/k\epsilon.$$

Elimination of $\mathbf{H}$ from (1) gives the wave equation

$$\nabla^2\mathbf{E}+K^2\mathbf{E}=\nabla\nabla\cdot\mathbf{E}-\mu^{-1}\nabla\mu\times(\nabla\times\mathbf{E}), \tag{3}$$

with the squared propagation constant

$$K^2=k^2\mu\epsilon'. \tag{4}$$

Equation (2) can again be obtained by taking the divergence of the wave equation.

Following reference (1), a Green's function

$$F(\mathbf{r},\mathbf{r}')=F(\mathbf{r}',\mathbf{r})=-(4\pi\rho)^{-1}e^{iS(\mathbf{r},\mathbf{r}')} \tag{5}$$

will be considered, where

$$\rho=|\mathbf{r}-\mathbf{r}'|;$$

the phase is assumed to have the limits

$$\lim_{\mathbf{r}'\rightarrow\mathbf{r}}\rho^{-1}S(\mathbf{r},\mathbf{r}')=C(\mathbf{r}),\qquad \lim_{r\rightarrow\infty}\nabla S=k\mathbf{n}+O(r^{-1});\quad \mathbf{r}=\mathbf{n}r. \tag{6}$$

This Green's function satisfies the differential equation

$$\nabla^2F+(\nabla S)^2F=\delta(\mathbf{r}-\mathbf{r}')+iF\rho^2\nabla\cdot(\rho^{-2}\nabla S). \tag{7}$$

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[1] D. S. Saxon and L. I. Schiff, Nuovo cimento **6**, 614 (1957).

[2] W. M. Brown, Ph.D. thesis, Department of Physics, University of California, Los Angeles, April, 1959 (unpublished); D. S. Saxon, IRE Transactions on Antennas and Propagation, Vol. AP-7, Special supplement, p. S320 (1959).

[3] L. I. Schiff, Phys. Rev. **103**, 443 (1956); **104**, 1481 (1956).

In order to apply Green's theorem to F and to the components of $\mathbf{E}$, we have to specify boundary conditions for $\mathbf{E}$ as $r\to\infty$. Anticipating later results, these must be assumed differently from the usual case, namely,

$$\lim_{r\to\infty}\mathbf{E}(\mathbf{r})=\mathbf{E}_0 e^{i[\mathbf{k}_0\cdot\mathbf{r}+\delta_0(\mathbf{r})]}+(1/r)e^{ikr}\mathbf{A}(\mathbf{k}_0,\mathbf{k}). \tag{8}$$

The second term is the usual spherical wave scattered in the direction

$$\mathbf{n}=\mathbf{k}/k,$$

but the first term is a plane wave incident along the direction

$$\mathbf{n}_0=\mathbf{k}_0/k$$

with additional phase

$$\delta_0(\mathbf{r})=-k\mathbf{n}_0\cdot\mathbf{r}+\lim_{r'\to\infty}[S(\mathbf{r},-\mathbf{n}_0 r')-kr']. \tag{9}$$

In the following, we shall, with reference 1, always consider the special choice for the phase

$$S(\mathbf{r},\mathbf{r}')=\int_0^\rho K(\mathbf{r}'+\hat{\rho}s)ds=\int_0^\rho K(\mathbf{r}-\hat{\rho}s)ds, \tag{10}$$

with

$$\boldsymbol{\varrho}=\mathbf{r}-\mathbf{r}',\quad \hat{\rho}=\boldsymbol{\varrho}/\rho,$$

corresponding to propagation along straight line paths with WKB phase. This is motivated by reasons of mathematical simplicity. It leads to a phase shift

$$\delta_0(\mathbf{r})=\int_0^\infty [K(\mathbf{r}-\mathbf{n}_0 s)-k]ds, \tag{11}$$

which modifies the incoming plane wave in the boundary condition, Eq. (8). This plane wave is not a solution of the wave equation, even in the limit $r\to\infty$; it satisfies in this limit

$$(\nabla^2+k^2)\mathbf{E}_0 e^{i(\mathbf{k}_0\cdot\mathbf{r}+\delta_0)}=\mathbf{E}_0 e^{i(\mathbf{k}_0\cdot\mathbf{r}+\delta_0)}[-2\mathbf{k}_0\cdot\boldsymbol{\nabla}\delta_0-(\boldsymbol{\nabla}\delta_0)^2+i\nabla^2\delta_0]. \tag{12}$$

[The scattered wave in Eq. (8), of course, is a solution of the wave equation to order r^{-1}.] Furthermore, the divergence equation (2), applied asymptotically to the boundary conditions of Eq. (8) and using $\mathbf{n}_0\cdot\mathbf{E}_0=0$, gives

$$\lim_{r\to\infty}\boldsymbol{\nabla}\cdot\mathbf{E}\equiv ik\left[\mathbf{E}_0 e^{i(\mathbf{k}_0\cdot\mathbf{r}+\delta_0)}\cdot\frac{\boldsymbol{\nabla}\delta_0}{k}+\frac{e^{ikr}}{r}\mathbf{n}\cdot\mathbf{A}\right]=0. \tag{13}$$

In the asymptotic limit, however, the phase shift $\delta_0(\mathbf{r})$ vanishes everywhere except in a cone of opening angle $\cong(R/r)$ around the forward direction $\mathbf{n}_0$, where R is the dimension of the scatterer. That means $\delta_0(\mathbf{r})$ appears only if *exact* forward scattering, $\mathbf{n}\equiv\mathbf{r}/r=\mathbf{n}_0$, is considered. In all other directions, our asymptotic solution is of the usual form and does satisfy the wave equation, and Eq. (13) also shows that the scattering amplitude $\mathbf{A}$ is polarized transversely to the propagation direction $\mathbf{n}$ everywhere except in the exact forward direction.

The differential scattering cross section is defined by

$$\frac{d\sigma}{d\Omega}=\lim_{r\to\infty}r^2\frac{\langle S_{\text{scatt}}\rangle}{\langle S_{\text{incid}}\rangle}, \tag{14}$$

where the energy flow $\langle\mathbf{S}\rangle$ is given by the real part of the complex Poynting vector, $\frac{1}{2}\operatorname{Re}(\mathbf{E}\times\mathbf{H}^*)$, and $\langle\ \rangle$ means time average. Considering the radial energy flow for S_{scatt}, we obtain

$$d\sigma/d\Omega=|\mathbf{A}_{\text{tr}}|^2/E_0^2,\quad \mathbf{A}_{\text{tr}}=\mathbf{A}-\mathbf{n}\mathbf{A}\cdot\mathbf{n}. \tag{15}$$

From a practical viewpoint, the total cross section can be taken as

$$\sigma_t'=\lim_{\eta\to+0}\int_\eta^\pi \sin\theta d\theta\int_0^{2\pi}d\varphi\frac{d\sigma}{d\Omega}, \tag{16}$$

if determined by measuring the scattering into all directions. A transmission experiment would, however, yield a different total cross section $\sigma_t=\int(d\sigma/d\Omega)d\Omega$ (no limit $\eta\to 0$), due to an additional forward component which, by interference with the forward part of the differential scattering, produces the geometrical shadow in the short-wavelength limit. The usual optical theorem,

$$\sigma_t=(4\pi/k)\operatorname{Im}(\mathbf{E}_0\cdot\mathbf{A}(\mathbf{k}_0,\mathbf{k}_0)/E_0^2), \tag{17}$$

does not hold any more in the present case: On the one hand, we have modified the boundary conditions in the forward direction; on the other hand, absorption is introduced by the conductivity σ. From Poynting's theorem,[4] we get instead

$$\sigma_t=\frac{4\pi}{k}\operatorname{Im}\frac{\mathbf{E}_0\cdot\mathbf{A}(\mathbf{k}_0,\mathbf{k}_0)}{E_0^2}+\frac{Q}{\frac{1}{2}E_0^2}-\lim_{r\to\infty}r^2\int\left[\mathbf{n}_0\cdot\mathbf{n}(e^{-2\operatorname{Im}\delta_0}-1)+e^{-2\operatorname{Im}\delta_0}\mathbf{n}_0\cdot\frac{\boldsymbol{\nabla}\operatorname{Re}\delta_0}{k}\right]d\Omega, \tag{18}$$

where Q is the power absorbed inside the scatterer—in the case of ϵ' complex, μ real, it is

$$Q=-\tfrac{1}{2}\int\sigma|\mathbf{E}|^2d\tau. \tag{19}$$

Green's theorem, applied to F and the components of $\mathbf{E}$, and taking account of (3), (7), (6), and the boundary condition (8), leads to an integral equation

[4] J. A. Stratton, *Electromagnetic Theory* (McGraw-Hill Book Company, Inc., New York, 1941), p. 458.

for $\mathbf{E}(\mathbf{r})$:

$$\mathbf{E}(\mathbf{r})=\mathbf{E}_0e^{i[\mathbf{k}_0\cdot\mathbf{r}+\delta_0(\mathbf{r})]}+\int d\tau'\,\mathbf{E}(\mathbf{r}')F(\mathbf{r},\mathbf{r}')\bigg\{[\boldsymbol{\nabla}'S(\mathbf{r},\mathbf{r}')]^2-K^2(\mathbf{r}')-i\rho^2\boldsymbol{\nabla}'\cdot\bigg[\frac{\boldsymbol{\nabla}'S(\mathbf{r},\mathbf{r}')}{\rho^2}\bigg]\bigg\}$$
$$+\int d\tau'[1-\epsilon'(\mathbf{r}')]\mathbf{E}(\mathbf{r}')\cdot\boldsymbol{\nabla}'\boldsymbol{\nabla}'F(\mathbf{r},\mathbf{r}')$$
$$+\int d\tau'\frac{\mu(\mathbf{r}')-1}{\mu(\mathbf{r}')}[\boldsymbol{\nabla}'F(\mathbf{r},\mathbf{r}')]\times[\boldsymbol{\nabla}'\times\mathbf{E}(\mathbf{r}')]$$
$$+k^2\int d\tau'\,\epsilon'(\mathbf{r}')[\mu(\mathbf{r}')-1]\mathbf{E}(\mathbf{r}')F(\mathbf{r},\mathbf{r}'),\quad(20)$$

where partial integrations have been used in the last three integrals. If the limit $r\rightarrow\infty$ is taken, all the integrals become $O(r^{-1})$, which suggests the use of the modified boundary condition (8). Following the techniques of reference 1, we find in this limit

$$\mathbf{A}(\mathbf{k}_0,\mathbf{k})=\frac{1}{4\pi}\bigg\{\int d\tau\; e^{i[\delta_+(\mathbf{r})-\mathbf{k}\cdot\mathbf{r}]}[K(\mathbf{r})-k]^2\mathbf{E}(\mathbf{r})$$
$$+2k\mathbf{E}_0\int d\tau\; e^{i[\mathbf{q}\cdot\mathbf{r}+\delta_+(\mathbf{r})]}[K(\mathbf{r})-k]$$
$$+i\int d\tau\; e^{i\delta_+(\mathbf{r})}[K(\mathbf{r})-k]\nabla^2\int_z^\infty e^{-i\mathbf{k}\cdot\mathbf{r}'}\mathbf{E}_{\mathrm{sc}}(\mathbf{r}')dz'$$
$$+\int d\tau[\epsilon'(\mathbf{r})-1]\mathbf{E}(\mathbf{r})\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}e^{i[\delta_+(\mathbf{r})-\mathbf{k}\cdot\mathbf{r}]}$$
$$-\int d\tau\frac{\mu(\mathbf{r})-1}{\mu(\mathbf{r})}[\boldsymbol{\nabla}e^{i[\delta_+(\mathbf{r})-\mathbf{k}\cdot\mathbf{r}]}]\times[\boldsymbol{\nabla}\times\mathbf{E}(\mathbf{r})]$$
$$-k^2\int d\tau\;\epsilon'(\mathbf{r})[\mu(\mathbf{r})-1]e^{i[\delta_+(\mathbf{r})-\mathbf{k}\cdot\mathbf{r}]}\mathbf{E}(\mathbf{r})\bigg\}.\quad(21)$$

where $\mathbf{q}=\mathbf{k}_0-\mathbf{k}$, the z axis is parallel to $-\mathbf{n}$, $\mathbf{r}'=(xyz')$,

$$\mathbf{E}_{\mathrm{sc}}(\mathbf{r})=\mathbf{E}(\mathbf{r})-\mathbf{E}_0e^{i\mathbf{k}_0\cdot\mathbf{r}},$$

and

$$\delta_+(\mathbf{r})=\int_0^\infty[K(\mathbf{r}+\mathbf{n}s)-k]ds.\quad(22)$$

The first term in an iteration treatment of Eq. (20) is expected to give a good approximation of the scattering amplitude provided that the magnitudes of ϵ', μ, and $\epsilon'\mu$ do not differ too much from unity. This leads to

$$\mathbf{A}(\mathbf{k}_0,\mathbf{k})\cong\frac{1}{4\pi}\bigg\{\mathbf{E}_0\int d\tau[K(\mathbf{r})-k]^2e^{i[\mathbf{q}\cdot\mathbf{r}+\delta_0(\mathbf{r})+\delta_+(\mathbf{r})]}$$
$$+2k\mathbf{E}_0\int d\tau[K(\mathbf{r})-k]e^{i[\mathbf{q}\cdot\mathbf{r}+\delta_+(\mathbf{r})]}$$
$$+i\mathbf{E}_0\int d\tau[K(\mathbf{r})-k]e^{i\delta_+(\mathbf{r})}\nabla^2\int_z^\infty e^{i\mathbf{q}\cdot\mathbf{r}'}[e^{i\delta_0(\mathbf{r}')}-1]dz'$$
$$+\int d\tau[\epsilon'(\mathbf{r})-1]e^{i[\mathbf{k}_0\cdot\mathbf{r}+\delta_0(\mathbf{r})]}\mathbf{E}_0\cdot\boldsymbol{\nabla}\boldsymbol{\nabla}e^{i[\delta_+(\mathbf{r})-\mathbf{k}\cdot\mathbf{r}]}$$
$$-\int d\tau\frac{\mu(\mathbf{r})-1}{\mu(\mathbf{r})}[\boldsymbol{\nabla}e^{i[\delta_+(\mathbf{r})-\mathbf{k}\cdot\mathbf{r}]}]\times[\boldsymbol{\nabla}\times\mathbf{E}_0e^{i[\mathbf{k}_0\cdot\mathbf{r}+\delta_0(\mathbf{r})]}]$$
$$-k^2\mathbf{E}_0\int d\tau\;\epsilon'(\mathbf{r})[\mu(\mathbf{r})-1]e^{i[\mathbf{q}\cdot\mathbf{r}+\delta_0(\mathbf{r})+\delta_+(\mathbf{r})]}\bigg\}.\quad(23)$$

Besides the condition of applicability mentioned above, one must also assume that $\epsilon'(\mathbf{r})$, $\mu(\mathbf{r})$ vary slowly over one wavelength (for the assumption of straight-line propagation), and that $kR\gg1$, as seen from the conditions of validity of reference 1; in detail, these are

$$\begin{aligned}1&\gg(kR)^{-1}\quad\text{if}\quad\zeta kR\lesssim1,\\1&\gg\zeta^2kR\quad\;\;\text{if}\quad\zeta kR\gtrsim1,\end{aligned}\quad(24)$$

where $\zeta=(\mu\epsilon')^{1/2}-1$ or $\epsilon'(\mu-1)$.

An exact solution would satisfy $\mathbf{n}\cdot\mathbf{A}=0$, from Eq. (13) (except in the forward direction). After the iteration leading to the approximation (23), there is no reason for this still to hold. Deviations from transversality of $\mathbf{A}$ cannot, however, be taken seriously, but are just an indication of the shortcomings of the theory. It will be shown in the following that both large-angle and small-angle limits of Eq. (23) lead to transverse scattering amplitudes.

I. LARGE-ANGLE APPROXIMATION

As shown in reference 1, this consists in an expansion in powers of k^{-1} and q_z^{-1}, where $q_z=k(1-\mathbf{n}\cdot\mathbf{n}_0)$. It leads to the exceedingly simple result

$$\mathbf{A}(\mathbf{k}_0,\mathbf{k})\cong\mathbf{n}\times(\mathbf{n}\times\mathbf{E}_0)I_e+\mathbf{n}\times(\mathbf{n}_0\times\mathbf{E}_0)I_m,\quad(25)$$

$$\begin{aligned}I_e&=-\frac{k^2}{4\pi}\int d\tau(\epsilon'-1)e^{i(\mathbf{q}\cdot\mathbf{r}+\delta_0+\delta_+)},\\I_m&=-\frac{k^2}{4\pi}\int d\tau\frac{\mu-1}{\mu}e^{i(\mathbf{q}\cdot\mathbf{r}+\delta_0+\delta_+)},\end{aligned}\quad(26)$$

manifestly transversely polarized, and in agreement with a formula, written in a different way, of the second

reference 3 (to the extent[5] that μ^{-1} can be set $=1$ in I_m). The differential cross section becomes, with $\hat{E}_0=\mathbf{E}_0/E_0$,

$$d\sigma/d\Omega=[1-(\mathbf{n}\cdot\hat{E}_0)^2]|I_e|^2+2\mathbf{n}\cdot\mathbf{n}_0\,\mathrm{Re}I_e{}^*I_m+[(\mathbf{n}\cdot\mathbf{n}_0)^2+(\mathbf{n}\cdot\hat{E}_0)^2]|I_m|^2. \quad (27)$$

Conditions of applicability are[1]

$$\begin{aligned} &1\gg[kR\sin^2(\theta/2)]^{-1} && \text{if } kR\zeta\lesssim 1,\\ &1\gg\text{max of } \zeta^2 kR \text{ or } \zeta\sin^{-2}(\theta/2) && \text{if } kR\zeta\gtrsim 1, \end{aligned} \quad (28)$$

where $\cos\theta=\mathbf{n}\cdot\mathbf{n}_0$. For $\mu=1$, Eqs. (25) and (26) also agree with the results of Brown.[2]

II. SMALL-ANGLE APPROXIMATION

This requires essentially[1]

$$\theta\ll(kR)^{-1/2}, \quad (29)$$

and Eq. (23) may be simplified in the following manner: The second term may be shown to be the leading term, and upon replacement of $\exp(iq_z z)$ by unity, gives the small-angle result

$$\mathbf{A}(\mathbf{k}_0,\mathbf{k})\cong\frac{ik}{2\pi}\mathbf{E}_0\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\; e^{i(q_x x+q_y y)} \times\left\{1-\exp\left[i\int_{-\infty}^{\infty}[K(\mathbf{r})-k]dz\right]\right\}, \quad (30)$$

which agrees with Schiff's[3] result if one still remembers that ϵ',μ are close to unity, so that $K-k\cong(k/2)(\epsilon'\mu-1)$; but again, Eq. (30) may be more accurate if the approximation is valid beyond sufficiency limits. The polarization of the small-angle scattering amplitude is seen to be transverse.

The remaining terms of Eq. (23) are negligible in comparison with (30): the third term by the arguments of reference 1, where also the conditions for this to happen are stated (they can be taken over for the vector case if one replaces $U\rightarrow\zeta k^2$). All other terms taken together can be shown to be of order $\zeta^2 kR$ relative to (30), and according to Eq. (24), this is negligible.

In conclusion, we state that the Saxon-Schiff theory has been applied to the scattering of electromagnetic waves from a weak scatterer of complex dielectric constant and permeability in the case $kR\gg 1$ (R being the dimension of the scatterer); the results of the large- and small-angle limits can be transformed into expressions obtained earlier, but may be more useful in the form obtained by us if the theory should be valid beyond the sufficiency limits. The large- and small-angle scattering amplitudes are transversely polarized, as demanded by the wave equation.

Finally, we wish to comment on the two-dimensional case, i.e., ϵ' and μ depend only on two coordinates (taken as y, z), and so do the fields. The boundary condition (8) then gets replaced by

$$\lim_{r\to\infty}\mathbf{E}(\mathbf{r})=\mathbf{E}_0 e^{i[\mathbf{k}_0\cdot\mathbf{r}+\delta_0(\mathbf{r})]}+r^{-1/2}e^{ikr}\mathbf{A}(\mathbf{k}_0,\mathbf{k}) \quad (8')$$

(we set $x=0$ always, and the vectors $\mathbf{k}_0,\mathbf{k}$ are assumed parallel to the $x=0$ plane), and the appropriate modified Green's function, instead of (5), has to be taken as

$$F(\mathbf{r},\mathbf{r}')=-\tfrac{1}{4}iH_0{}^{(1)}[S(\mathbf{r},\mathbf{r}')], \quad (5')$$

(Hankel function of zero order), in which we also adopt the S from Eq. (10). Using the same procedure as before, we obtain completely analogous results: The new $A(\mathbf{k}_0,\mathbf{k})$ is again given by the previous equations (21), (23), (25), and (30), except that $\int dx$ is absent, and a factor

$$e^{i\pi/4}(2\pi/k)^{1/2} \quad (31)$$

has to be added. The "cross section," by replacing $r^2 d\Omega\rightarrow rd\varphi$ in Eq. (14), has the dimension of length; then $d\sigma/d\varphi$ is again given by the expression of Eq. (15).

My sincere thanks are due to Dr. Vaughan Weston for suggesting this problem and for discussion and encouragement, and to Professor K. M. Siegel for his interest. I also wish to thank Professor L. I. Schiff for his reading of the manuscript.

APPENDIX

We wish to compare numerically the results of the Saxon-Schiff theory for scattering by a homogeneous sphere with those of the exact Mie theory,[6,7] in order to get an idea of the applicability of the method. We chose to check the large-angle expression (25), so as to include the radar cross section. Designating now the radius of the sphere by R, and calling $kR=\rho$, we have for the differential scattering cross section (reference 7, p. 127)

$$d\sigma/R^2 d\Omega=(1/\rho^2)[i_1(\theta)\sin^2\varphi+i_2(\theta)\cos^2\varphi] \quad (\text{A1})$$

for polarized incident waves, φ being the azimuth between the $(\mathbf{n}_0,\mathbf{E}_0)$ plane and the $(\mathbf{n}_0,\mathbf{n})$ plane. The functions i_1, i_2 have been tabulated.[8] Unpolarized radiation gives

$$d\sigma/R^2 d\Omega=[i_1(\theta)+i_2(\theta)]/2\rho^2. \quad (\text{A2})$$

From (27), the Saxon-Schiff expressions for i_1, i_2 are

[5] In the case that the Saxon-Schiff approximation is valid beyond the limits *sufficient* for applicability, stated throughout this paper [as is often true for the Born approximation (see reference 2)], I_m is considered to be more accurate than the corresponding formula of reference 3.

The Born approximation result is obtained by setting $\delta_0\equiv\delta_+\equiv 0$ in (25) and (26).

[6] J. A. Stratton, reference 4, p. 563.

[7] H. C. van de Hulst, *Light Scattering by Small Particles* (John Wiley & Sons, Inc., New York, 1957).

[8] E.g., A. N. Lowan, National Bureau of Standards, Applied Mathematics Series 4 (U. S. Government Printing Office, Washington, D. C., 1948); graphs and a complete list of tabulations are given in reference 7, pp. 152, 167.

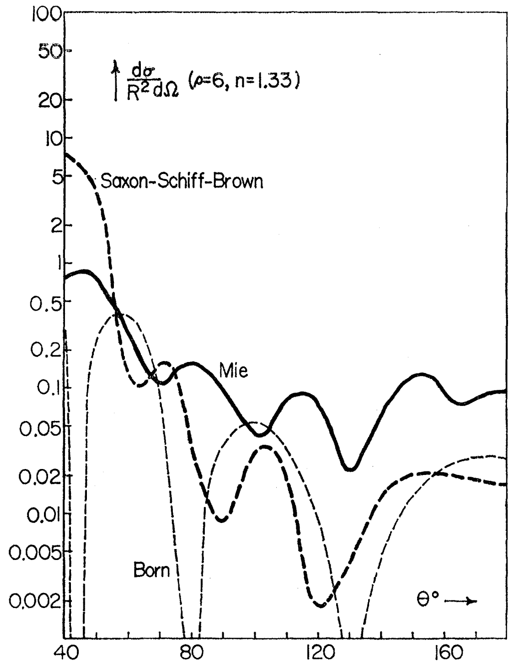

FIG. 1. Angular distribution of diffracted radiation from a homogeneous dielectric sphere in Mie theory, Saxon-Schiff-Brown, and Born approximation; $\rho=6$, $n=1.33$.

found to be

$$i_1=\rho^2|\epsilon'-1+[(\mu-1)/\mu]\cos\theta|^2|I|^2, \qquad \text{(A3a)}$$

$$i_2=\rho^2|(\epsilon'-1)\cos\theta+[(\mu-1)/\mu]|^2|I|^2, \qquad \text{(A3b)}$$

with

$$I=-\frac{k^2}{4\pi R}\int d\tau\, e^{i(\mathbf{q}\cdot\mathbf{r}+\delta_0+\delta_+)}, \qquad \text{(A3c)}$$

integrated over the sphere. The Born approximation gives now the result (j_1=spherical Bessel function)

$$|I|_{\text{Born}}{}^2=[\rho^2/4\sin^2(\tfrac{1}{2}\theta)]j_1{}^2(2\rho\sin\tfrac{1}{2}\theta). \qquad \text{(A4)}$$

In the Saxon-Schiff theory, we obtain for the phases, e.g.,

$$\delta_+(\mathbf{r})=(n-1)\rho\left[\left(1-\frac{r^2}{R^2}\left(1-\mathbf{n}\cdot\frac{\mathbf{r}}{r}\right)\right)^{1/2}-\frac{1}{R}\mathbf{n}\cdot\mathbf{r}\right],$$

where $n=(\epsilon'\mu)^{1/2}$. This leads to

$$I_{\text{SS}}=-\frac{\rho^2}{4\pi}\int_0^1 xdx\int_0^\pi \sin\theta' d\theta'\int_0^{2\pi} d\varphi'$$
$$\times\exp\{2i\rho x\sin\tfrac{1}{2}\theta\cos\theta'+i\rho(n-1)[(1-x^2\sin^2\beta_+)^{1/2}$$
$$+(1-x^2\sin^2\beta_-)^{1/2}+x(\cos\beta_+-\cos\beta_-)]\}, \qquad \text{(A5)}$$

with

$$\cos\beta_\pm=\pm\sin\tfrac{1}{2}\theta\cos\theta'+\cos\tfrac{1}{2}\theta\sin\theta'\cos\varphi'.$$

This could be integrated numerically, but we chose instead to evaluate it using a further approximation suggested by Brown.[2] This consists of two steps. First, in I, a partial integration is made with respect to the variable $d(\hat{q}\cdot\hat{r})$ where $\hat{q}=\mathbf{q}/q$, $\hat{r}=\mathbf{r}/r$, and the integral term is neglected, which does not lead to larger errors than already contained in the Saxon-Schiff method. Then the azimuthal integration can be done immediately, and the resulting integral

$$I_{\text{SSB}}=\frac{ik^2}{2qR}\int_{-R}^{R} rdr\,\exp i[qr+\delta_0(\hat{q}r)+\delta_+(\hat{q}r)], \qquad \text{(A6)}$$

which in its explicit form reads

$$I_{\text{SSB}}=\frac{-\rho}{2\sin\frac{1}{2}\theta}\frac{1}{2i}\int_{-1}^{1} xdx\,\exp\{2i\rho[nx\sin\tfrac{1}{2}\theta$$
$$+(n-1)(1-x^2\cos^2\tfrac{1}{2}\theta)^{1/2}]\}, \qquad \text{(A7)}$$

could also be evaluated numerically, but is obtained (second step) as an asymptotic expansion in ρ^{-1}, again by partial integration:

$$|I|_{\text{SSB}}{}^2=(\rho^2/4\sin^2\tfrac{1}{2}\theta)\left[j_1{}^2(2n\rho\sin\tfrac{1}{2}\theta)\right.$$
$$\left.+\left(\frac{n-1}{n}\cot^2(\tfrac{1}{2}\theta)\right)^2\frac{1+\cos^2(2n\rho\sin\frac{1}{2}\theta)}{(2n\rho\sin\frac{1}{2}\theta)^2}\right]. \qquad \text{(A8)}$$

In obtaining this result, we have, for simplicity, also assumed that n is real.

In Figs. 1 and 2, we present the values of the un-

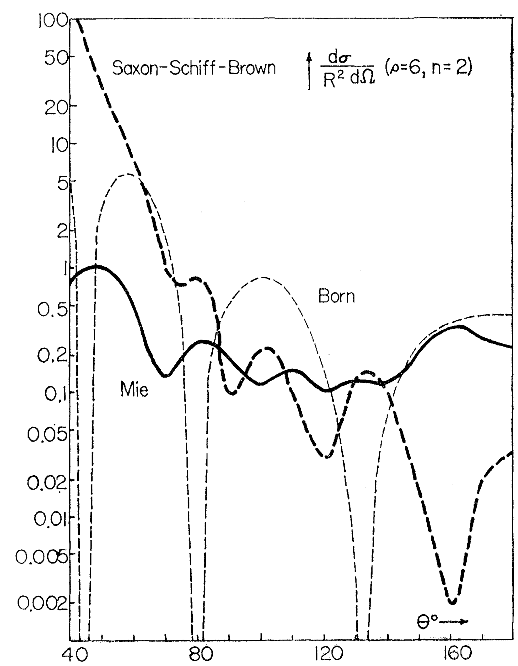

FIG. 2. Angular distribution of diffracted radiation from a homogeneous dielectric sphere in Mie theory, Saxon-Schiff-Brown, and Born approximation; $\rho=6$, $n=2$.

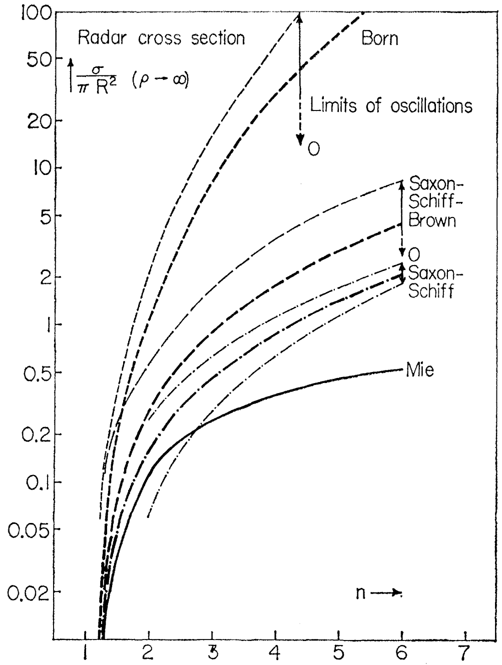

FIG. 3. Radar cross section of a homogeneous dielectric sphere (asymptotically for $\rho \to \infty$) vs index of refraction n, in Mie theory, Saxon-Schiff, Saxon-Schiff-Brown, and Born approximation.

polarized differential cross section (A2) from the exact theory,[8] and from the Saxon-Schiff-Brown (A8) and the Born approximation (A4) for homogeneous spheres with $\mu=1$, real ϵ, for $\rho=6$ and $n=1.33$ and 2. The conditions of applicability, $\rho \gg 1$ and n close to unity, are only partly satisfied, and furthermore, the condition that ϵ', μ vary slowly over one wavelength is clearly violated for a body with sharp boundaries. Nevertheless, it can be seen that there is good qualitative agreement between the approximate and the exact angular distribution, as the number and position of the diffraction maxima and minima are reproduced fairly well. Some large deviations occur for the larger of the two values of n considered, and the backward scattering is never given accurately. However, the improvement over the Born approximation is striking, the oscillations of the latter showing no connection with reality.

Lastly, we evaluated the radar cross section ($\theta=180°$) in the asymptotic limit $\rho \to \infty$. In the Born approximation, we find (always for $\mu=1$, n real)

$$(\sigma/\pi R^2)_{\mathrm{Born}} \to \tfrac{1}{8}(n^2-1)^2(1+\cos 4\rho); \tag{A9}$$

the Saxon-Schiff result can in this case be obtained exactly from (A5)

$$\left(\frac{\sigma}{\pi R^2}\right)_{\mathrm{SS}} \to \frac{(n^2-1)^2}{16n^2}\left(1+\frac{1}{(2n-1)^2} + \frac{2}{2n-1}\cos(4n\rho)\right), \tag{A10}$$

which thus permits an estimate of the further errors introduced by using Brown's approximation

$$(\sigma/\pi R^2)_{\mathrm{SSB}} \to [(n^2-1)^2/8n^2][1+\cos(4n\rho)]. \tag{A11}$$

The exact result is simply given by the Fresnel coefficient, since $\rho \to \infty$ may be interpreted as a large sphere to which geometrical optics is applicable[7]:

$$(\sigma/\pi R^2)_{\mathrm{Mie}} \to [(n-1)/(n+1)]^2. \tag{A12}$$

Figure 3 shows a comparison between these results. They all agree more or less for n close to unity, and the Saxon-Schiff, Brown, and Born approximations give curves which deviate successively more and more from the exact expression. We have also indicated the limits of the oscillation given by the cosine terms in (A9) through (A11).