Complex Angular Momenta and Unitarity in Crossed Channels*

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(Received July 26, 1962)

Regge poles are assumed to exist and the further singularities in the Regge plane resulting from the effect of the separate terms in the unitary sum in the crossed channels are investigated. Three families of singularities are found. However, the equations of the singular curves are independent of the number of particles in the intermediate state corresponding to the term in the unitary sum being considered. Consequently, no firm conclusion can be made about the behavior of the complete amplitude, owing to the possibility of cancellations.

1. INTRODUCTION

AN important feature of a full relativistic theory which is not found in potential models is the requirement of unitarity in crossed channels. When scattering amplitudes are considered as functions of invariants, this requirement leads to the complicated analytic structure given by the Landau equations.[1] It is of interest therefore to investigate as far as possible the corresponding consequences for representations using complex angular momenta.[2]

The Froissart definition[3] of the Regge amplitude is

$$a_{\pm}(l,s)=\int_{z_0}^{\infty}\rho_{\pm}(s,z)Q_l(z)dz, \tag{1}$$

where ρ_+ is the symmetrical and ρ_- is the antisymmetrical combination of the discontinuities in the crossed channels, z is the cosine of the scattering angle, and z_0 is the value of that cosine corresponding to the lowest threshold in the crossed channels. The singularities of $Q_l(z)$ at $z=\pm 1$ give the familiar partial-wave singularities of $a(l,s)$ that occur at fixed values of s for all l. Singularities that are functions of l arise from the behavior of the integral (1) at its infinite limit. This behavior is influenced by subtle cancellations that are not easy to analyze.

2. UNITARY INTEGRALS

The discontinuities appearing in $\rho_{\pm}$ are given by sums of terms with each sum representing unitarity for an n-particle intermediate state,

$$\rho_n=\int f_n{}^* f_n d\Omega_n, \tag{2}$$

where the integral is over n-particle phase space. Since we are concerned with the behavior of (1) at infinity, we are concerned with an infinite sum of terms of type (2).

For any one of these terms we can ask the following question. If Regge poles are assumed to exist in all scattering amplitudes f_n, it is possible to find the resulting behavior of ρ_n for large z; what, then, are the resulting singularities of $a_{\pm}(l,s)$? This question is the analog in the Regge plane of the question: What singularities in invariant space are generated by putting normal thresholds in crossed channels into the unitary integrals?[1] It should be emphasized that the type of iteration involved here is concerned with the study of the internal consistency of a theory and is not to be thought of as involving an actual perturbation expansion.

We shall assume, therefore, that in a general scattering amplitude there occur Regge poles—such as the pole represented by Fig. 1—that give a contribution

$$\beta(s;s_i)\frac{P_{\alpha(s)}\left(1+\dfrac{2t}{s-4m^2}\right)}{\sin\pi\,\alpha(s)}, \tag{3}$$

where $s=(q_1+q_2)^2$, $t=(p_1+q_1)^2$, and the s_i are the remaining invariants necessary to specify the configuration. For brevity we have omitted the necessary symmetrization or antisymmetrization in (3). Other Regge poles for which different kinds of partitions of the external momenta are made may be expected to occur, but they are not considered here.

The phase-space integrals occurring in (2) may best be expressed in terms of integrals over invariants. A convenient choice of these invariants is shown in the vector diagram, Fig. 2. $ABCD$ represents the four

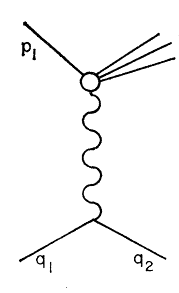

FIG. 1. A postulated Regge pole.

* Work done under the auspices of the U. S. Atomic Energy Commission.

† On leave of absence from the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, and Trinity College, Cambridge, England.

[1] J. C. Polkinghorne, Nuovo cimento 23, 360 (1962); 25, 901 (1962); H. P. Stapp, Phys. Rev. 125, 2139 (1962).

[2] T. Regge, Nuovo cimento 14, 951 (1958); 18, 947 (1960); A. Bottino, A. M. Longoni, and T. Regge, *ibid.* 23, 954 (1962).

[3] M. Froissart, in "Proceedings of the La Jolla Conference on Weak and Strong Interactions, 1961" (unpublished).

external lines with $BD^2=t$, $AC^2=s$. BO_1, O_1O_2, $\cdots$ represent the momenta of the n particles in the intermediate state. The squares of the lengths of the remaining lines in the diagram, s_i, s_i', λ_i, then serve to determine the configuration, for when they are known the simplexes $ABCDO_1$, AO_1CDO_2, $\cdots$ can be successively constructed. The Jacobian J^{-1} of the transformation from the integral over momenta to the integral over invariants is given by

$$J=\Delta_1\cdot\Delta_2\cdot\ \cdots, \tag{4}$$

where Δ_i is the four volume of the ith simplex. The boundary of the region of integration is given by $J=0$. J^{-1} is always singular on this boundary but it can be shown by suitable limiting arguments that this does not cause trouble. Finally, we notice that the only factor in J that contains t is Δ_1, which is linear in t for large t.

3. SINGULARITIES

We now substitute expressions of the form (3) for f in (2) in order to determine the singularities corresponding to these Regge poles; that is, we determine the singularities of

$$\int_{t_0}^{\infty}\frac{2dt}{s-4m^2}\int\frac{ds_ids_i'd\lambda_i}{J}Q_l\left(1+\frac{2t}{s-4m^2}\right)\times\left[\frac{\beta(s_1')P_{\alpha(s_1')}\left(1+\dfrac{2t}{s_1'-4m^2}\right)}{\sin\pi\,\alpha(s_1')}\right]^*\times\left[\frac{\beta(s_1)P_{\alpha(s_1)}\left(1+\dfrac{2t}{s_1-4m^2}\right)}{\sin\pi\,\alpha(s_1)}\right]. \tag{5}$$

All the integrations occurring in (5) can be analyzed by standard methods[4] except for the t integration. This latter becomes formally divergent for *all* sufficiently small values of Re l. In fact, however, one knows that this is a formal difficulty and that the study of cancellations due to oscillations at $t=\infty$ will permit analytic continuation below this seeming barrier. In order to do this it is necessary to perform the t integration first for sufficiently large values of Re l and continue the result to other values of l. Examination of the asymptotic expansions of all the factors in (5) that have a dependence on t shows that the resulting function may have singularities given by

$$l=\alpha^*(s_1')+\alpha(s_1)-n,\quad\text{for}\quad n=1,2,\cdots. \tag{6}$$

This result is obtained by subtracting a finite number of terms in each asymptotic expansion to leave a remainder which has an enlarged domain of obvious analyticity in l. The subtracted terms then give singularities given by (6).

The singularities of the complete integral (5) may now be found by looking for pinches or end points in the remaining integrations. Neglecting for the moment the effects at singularities of $\alpha(s)$, one sees that this is equivalent to finding the stationary values of

$$\alpha^*(s_1')+\alpha(s_1)-n,$$

either for unrestricted variations or for variations on the analytic manifold defining the boundary of the region of integration.[5] This latter must, of course, be evaluated with $t=\infty$, since the singularities (6) arise at infinity in the t integration.

It is important to notice that, because Cauchy's theorem allows the contours of integration to be distorted, the stationary values must be evaluated for general variations and not merely for variations within the physically real manifold of values of the invariants. Thus, simple arguments counting the maximum power of t in the physical region of integration are not adequate. This is just the familiar fact that one must choose paths of stationary phase when finding asymptotic behavior. For this reason it does not seem possible to assert that a contradiction occurs with a spin 1 resonance in the strip approximation[6] unless something is known about the behavior of $\alpha(t)$ for complex t.

Variations on the boundary of the region of integration can be discussed directly or their behavior can be linked with known results on the invariant space properties of singularities. From the latter point of view the same phase-space integrals are considered. They are usually expressed in terms of momenta rather than invariants in this case, but it is easy to take the invariant approach. In our problem the invariants

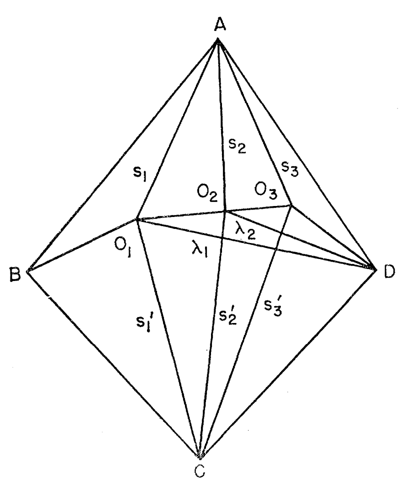

FIG. 2. Diagram to show the choice of invariants.

[4] J. C. Polkinghorne and G. R. Screaton, Nuovo cimento **15**, 289 (1960).

[5] Cf., J. C. Polkinghorne, *Brandeis Lecture Notes I, 1961,* (W. A. Benjamin, Inc., New York, 1962), Chap. 1.

[6] Geoffrey F. Chew, Steven C. Frautschi, and Stanley Mandelstam, Phys. Rev. **126**, 1202 (1962).

$s_2, s_3, \cdots, s_2', s_3', \cdots, \lambda_1, \lambda_2, \cdots$ do not have any singularities associated with them other than the singularities of J^{-1}. In diagrammatic language this means that the corresponding lines are contracted out. Therefore, the boundary manifold over which s_1 and s_1' are to vary in calculating the stationary values of (6) is obtained by putting $t=\infty$ in the Landau curve associated with Fig. 3. This gives

$$\lambda(s,s_1,s_1')\equiv s^2+s_1^2+s_1'^2-2ss_1-2s_1s_1'-2s_1's=0. \quad (7)$$

This curve is independent of the number of contracted lines in Fig. 3 joining A to B. Thus every term in the unitary sum yields *exactly* the same set of curves for these singularities. We call this set of singularities S_2 The set S_1 is the set corresponding to stationary values of (6) in the interior of the domain of integration. These are singularities at fixed l, independent of the value of s. Nothing can be said about them in detail without a knowledge of the form of $\alpha(s)$. Again they will be common to all the terms in the unitary sum.

A third set of singularities S_3 can be associated with the effect of singularities in $\alpha(s)$. If such a singularity occurs at $s=s_0$ then singularities of the integral occur at

$$l=\alpha^*(s_0')+\alpha(s_0)-n,$$
$$l=\alpha^*(s_0)+\alpha(s_0')-n, \quad (8)$$

where $s_0'(s)$ is given by

$$\lambda(s,s_0,s_0')=0. \quad (9)$$

A similar set of singularities occurs when s_0 is such that $\alpha(s_0)$ is an integer of correct parity to give a pole in a correctly symmetrized expression (3). Singularities of type S_3 arise from pinches involving one more singular surface than those of type S_2. We expect, therefore, that the former are switched on and off by the latter.[7]

So far we have considered singularities arising from the effect of the same Regge pole in both f_n and f_n^*. Of course, similar singularities are to be expected which combine two different Regge trajectories.

4. DISCUSSION

The singularity corresponding to S_2 with $n=1$ has been discussed independently by Amati, Fubini, and Stanghellini[8] for the elastic term in the unitary sum. However, we have found that an identical singularity can be associated with every other term in the sum.

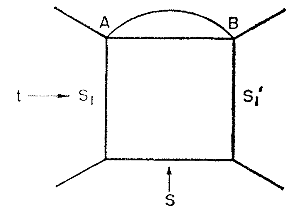

FIG. 3. Diagram giving equivalent Landau curve.

(This contrasts with the situation in invariant space,[1] where the different terms give different Landau curves, so that cancellations are not to be expected in that case.) Since all Regge phenomena are fraught with cancellation, it does not seem possible to make any firm conclusion from the study of separate terms in the unitary sum. We note, however, that this singularity has been proposed in the case of nucleon-nucleus scattering by Udgaonkar and Gell-Mann[9] on the basis of impulse-approximation calculations. The analysis of this paper does not, however, seem to suggest any obvious connection with the existence of anomalous thresholds.

If it were possible to obtain analytic properties from the separate terms of the unitary sum, then one might expect for self-consistency that the original Regge poles should themselves reappear as a result of the iterative analysis. The one way in which this could happen would be if the Pomeranchuk trajectory $\alpha_P(s)$ had a singularity at $s=0$ and also satisfied $\alpha_P(s)=1$. Then any other trajectory would reproduce itself by combining with α_P in an S_3-type singularity with $n=1$. Delicate conditions would also have to be satisfied to ensure that this singularity was still a pole. It is interesting to notice that Frye[10] has recently proposed a high-energy behavior of scattering amplitudes which seems consistent with this assumption. Such an assumption could also remove the difficulty of increasingly more violent asymptotic behavior found by means of successive iterations by Amati, Fubini, and Stanghellini.[8]

The actual occurrence of singularities in the *separate* terms of the unitary sum on specific Riemann sheets of the function seem to be most conveniently studied by explicit examination of the integrals and their asymptotic behaviors. The leading $n=1$ singularities are particularly easily found.[8]

ACKNOWLEDGMENT

It is a pleasure to thank Dr. David Judd for hospitality at the Lawrence Radiation Laboratory.

[7] Cf. M. Fowler, J. Math. Phys. **3**, 936 (1962).
[8] D. Amati, S. Fubini, and A. Stanghellini, Phys. Letters **1**, 29 (1962) and (to be published).
[9] B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962).
[10] Graham Frye, Phys. Rev. Letters **8**, 494 (1962).