and

$$W_z=\frac{1}{5\zeta_0}\left[1-\frac{5\pi}{2}\left(\frac{\hbar\omega_0}{2\zeta_0}\right)^{1/2}\frac{kT}{\zeta_0}\times\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{1/2}}\frac{\sin[(2\pi n\zeta_0/\hbar\omega_0)-\pi/4]}{\sinh[2n\pi^2kT/\hbar\omega_0]}-\frac{5}{12}\left(\frac{\hbar\omega_0}{2\zeta_0}\right)^2\right]. \quad (22)$$

III. ATTENUATION OF ACOUSTIC WAVES IN A LONGITUDINAL MAGNETIC FIELD

With the approximation $\omega\tau\ll1$, considered in this paper, the coefficient of ultrasonic attenuation in the case of propagation parallel to the applied magnetic field turns out to be the same as in the semiclassical limit to a high degree of accuracy. In fact, up to and including terms of the order of $(\hbar\omega_0/\zeta_0)^2$ we find no quantum effects similar to those discussed in Sec. II. This result differs radically from that found in the case in which $\omega\tau\gg1$. For example, in the latter situation, the coefficient of attenuation for $\omega_0\gg\omega$ experiences giant oscillations which have been described in detail elsewhere.[7] When $\omega\tau\approx1$, the expressions for the attenuation coefficients are unwieldy and we have not been able to obtain results of the simplicity of those exhibited in Eqs. (18)–(20). The giant oscillations predicted by Gurevich *et al.*[7] have been observed by Korolyuk and Prushchak[8] in a Zn sample where the ratio of the resistivity at liquid helium temperature to that at room temperature is 3×10^{-5} and at an ultrasonic frequency of 200 Mc/sec.

To summarize, there exists two types of oscillatory quantum effects in ultrasonic attenuation in metals as a function of an applied magnetic field. The first, which is described in Sec. II, is an oscillation in γ of small amplitude [proportional to $(\hbar\omega_0/2\zeta_0)^{3/2}$] superimposed on the ordinary semiclassical coefficient of attenuation. The second appears in the form of giant oscillations when the acoustic wave propagates in the direction of the magnetic field. The first type of oscillations become observable when $\omega_0\tau$ and ω_0/qv_0 are much larger than unity, while the second type is observable only if $\omega_0\gg\omega>\tau^{-1}$.

[7] V. L. Gurevich, V. G. Skobov, and Yu. A. Firsov, Soviet Phys.—JETP **13**, 552 (1961).
[8] A. P. Korolyuk and T. A. Prushchak, Soviet Phys.—JETP **14**, 1201 (1962).

PHYSICAL REVIEW VOLUME 128, NUMBER 6 DECEMBER 15, 1962

Simple Model for Nucleation around Dislocations*

C. ABRAHAM AND A. AHARONI
Department of Electronics, The Weizmann Institute of Science, Rehovoth, Israel
(Received July 25, 1962)

The effect of the mechanical stresses at a dislocation, via magnetostriction, on the theoretical nucleation field for magnetization reversal is approached by assuming a cylindrical region in which the magnetocrystalline anisotropy vanishes. The complete spectrum of eigenvalues is studied for this model, and it is found that the buckling mode yields the lowest eigenvalues for hard materials with reasonable size for the defective region. The turnover to the curling mode is at a radius of about 300 Å for MnBi and 550 Å for $BaFe_{12}O_{19}$. To obtain the observed value of nucleation field in $BaFe_{12}O_{19}$, the radius of the cylinder with $K=0$, should be about 350 Å, which seems plausible for a dislocation.

1. INTRODUCTION

IT has been suggested by Rathenau *et al.* that domain walls might start to nucleate at regions where the magnetocrystalline anisotropy constant is low, because of some structural imperfections.[1] This possibility has been studied for simple one-dimensional models,[2] giving rather encouraging results for hard materials.[3] These are extended here to three dimensions, giving a fuller description of the calculations mentioned in a recent review.[3]

Since measurements for hard materials are usually carried out on crystals in the form of thin plates, the calculation reported here assumes the material to be an infinite slab of width $2a$, perpendicular to the z axis, which is assumed to be an easy axis, and at which direction the external magnetic field is applied. The imperfection is assumed to be a region in the form of a cylinder of radius R, perpendicular to the slab, in which the magnetocrystalline anisotropy coefficient $\bar{K}$ vanishes. In cylindrical coordinates one has, thus, putting the z axis along the center line of the cylinder,

* This work will be included in a thesis by C. Abraham to be submitted to the Hebrew University, Jerusalem, in partial fulfilment of the requirements for a degree of Doctor of Philosophy.
[1] G. W. Rathenau, J. Smit, and A. L. Stuyts, Z. Physik **133**, 250 (1952).
[2] A. Aharoni, Suppl. J. Appl. Phys. **30**, 70S (1959); Phys. Rev. **119**, 127 (1960); C. Abraham and A. Aharoni, *ibid.* **120**, 1576 (1960).
[3] Amikam Aharoni, Revs. Modern Phys. **34**, 227 (1962).

$$\bar{K}(r)=0 \quad \text{for} \quad r\leq R, \qquad =K \quad \text{for} \quad r\geq R. \tag{1}$$

The differential equations for the nucleation problem follow from Brown's equation for equilibria[4] in the same way one derives the equation for an infinite cylinder[5] and are, in cylindrical coordinates, r, φ, z:

$$[\nabla'^2-t^{-2}-\pi S^2h]\alpha_r-2t^{-2}\partial\alpha_\varphi/\partial\varphi-\pi S\partial u/\partial t=0, \tag{2a}$$

$$[\nabla'^2-t^{-2}-\pi S^2h]\alpha_\varphi+2t^{-2}\partial\alpha_r/\partial\varphi-\pi St^{-1}\partial u/\partial\varphi=0, \tag{2b}$$

$$\nabla'^2u-2S[\partial\alpha_r/\partial t+t^{-1}(\alpha_r+\partial\alpha_\varphi/\partial\varphi)]=0, \tag{2c}$$

$$\text{for} \quad t^2<1, \; p^2<q^2; \tag{2d}$$

$$[\nabla'^2-t^{-2}-\pi S^2(h+g)]\alpha_r-2t^{-2}\partial\alpha_\varphi/\partial\varphi-\pi S\partial u/\partial t=0, \tag{3a}$$

$$[\nabla'^2-t^{-2}-\pi S^2(h+g)]\alpha_\varphi+2t^{-2}\partial\alpha_r/\partial\varphi-\pi St^{-1}\partial u/\partial\varphi=0, \tag{3b}$$

$$\nabla'^2u-2S[\partial\alpha_r/\partial t+t^{-1}(\alpha_r+\partial\alpha_\varphi/\partial\varphi)]=0, \tag{3c}$$

$$\text{for} \quad t^2>1, \; p^2<q^2; \tag{3d}$$

and

$$\nabla'^2u=0, \quad \text{for} \quad p^2>q^2. \tag{4}$$

Here

$$\nabla'^2=\partial^2/\partial t^2+t^{-1}\partial/\partial t+t^{-2}\partial^2/\partial\varphi^2+\partial^2/\partial p^2, \tag{5}$$

$$t=r/R, \quad p=z/R, \quad h=H/2\pi I_s, \quad S=RI_sA^{-1/2}, \quad u=U(2\pi)^{-1}A^{-1/2}, \quad g=K/\pi I_s^2, \quad q=a/R, \tag{6}$$

I_s is the saturation magnetization, A is the exchange constant, H is the magnetizing force *inside* the material, U is the potential of surface and volume charges, and α_r, α_φ are the direction cosines of the magnetization.

The boundary conditions are that

$$\partial\alpha_r/\partial p=\partial\alpha_\varphi/\partial p=0 \quad \text{on} \quad p=\pm q, \tag{7}$$

that α_r, α_φ, u and their derivatives are continuous on $t=1$, u and $\partial u/\partial p$ are continuous on $p=\pm q$.

The solution of this general set of equations is complicated. However, as will be seen in the following, one is usually interested only in large values of q. In the following section, we shall therefore give a rigorous solution for $q=\infty$. In Sec. 3 it will be shown that this solution is a good approximation for the case $q\gg1$. The discussion in Sec. 4 will relate the results to available experimental data. It will be seen that to fit the observed nucleation field in barium ferrite, the radius of the imperfect region should be 350 Å, which is not unreasonable for a dislocation.

2. INFINITE WIDTH

For $q=\infty$ one does not have to consider (4) and (7). The sets of Eqs. (2) and (3) can be solved separately using practically the same analytical solution as in the case of an infinite cylinder.[5] The variables are separated by writing

$$\alpha_r=A_r(t)\cos(kp-p_0)\cos(m\varphi-\varphi_0), \tag{8a}$$

$$\alpha_\varphi=A_\varphi(t)\cos(kp-p_0)\sin(m\varphi-\varphi_0), \tag{8b}$$

$$u=U_t(t)\cos(kp-p_0)\cos(m\varphi-\varphi_0), \tag{8c}$$

where m is an integer and k is a real number. The solution of (2) for the t-dependent part should involve functions which are regular for $t=0$, while the solution for (3) should involve functions which are regular for $t=\infty$. One therefore obtains for $t\leq1$

$$A_r-A_\varphi=\sum_{n=1}^{3}a_nJ_{m-1}(i\mu_nt), \tag{9a}$$

$$U_t=2iS\sum_{n=2}^{3}[a_n\mu_n/(k^2-\mu_n^2)]J_m(i\mu_nt), \tag{9b}$$

$$A_r+A_\varphi=\sum_{n=1}^{3}(-1)^{(1/2)n(n-1)}a_nJ_{m+1}(i\mu_nt), \tag{9c}$$

and for $t>1$

$$A_r-A_\varphi=\sum_{n=1}^{3}b_nH_{m-1}^{(1)}(i\nu_nt), \tag{10a}$$

$$U_t=2iS\sum_{n=2}^{3}[b_n\nu_n/(k^2-\nu_n)]H_m^{(1)}(i\nu_nt), \tag{10b}$$

$$A_r+A_\varphi=\sum_{n=1}^{3}(-1)^{(1/2)n(n-1)}b_nH_{m+1}^{(1)}(i\nu_nt), \tag{10c}$$

where J_m are Bessel's functions of the first kind and $H_m^{(1)}$ are Bessel's functions of the third kind, a_n and b_n are the 6 constants of integrations, while

$$\mu_1^2=k^2+\pi S^2h, \tag{11a}$$

$$\mu_{2,3}^2=k^2+\pi S^2(\tfrac{1}{2}h+1)\mp[2\pi k^2S^2+\pi^2S^4(\tfrac{1}{2}h+1)^2]^{1/2}, \tag{11b}$$

$$\nu_1^2=k^2+\pi S^2(h+g), \tag{11c}$$

$$\nu_{2,3}^2=k^2+\pi S^2[1+\tfrac{1}{2}(h+g)]\mp\{2\pi k^2S^2+\pi^2S^4[1+\tfrac{1}{2}(h+g)]\}^{1/2}. \tag{11d}$$

The functions $A_r\pm A_\varphi$, U_t and their derivatives with respect to t should be continuous at $t=1$. Using (9) and (10) this implies 6 linear relations between the 6 parameters a_n and b_n. A nonvanishing solution is therefore possible if and only if the determinant of the coefficients, $D_m(H,k,S,g)$, is zero. The determinants might, of course, be considered separately for the different integral values of m.

The calculation is rather simple for the case $m=0$. In this case D is a product of 2 determinants, one of which involves only the coefficients of A_φ and implies

$$i\mu_1J_0(i\mu_1)/J_1(i\mu_1)=i\nu_1H_0^{(1)}(i\nu_1)/H_1^{(1)}(i\nu_1). \tag{12}$$

[4] W. F. Brown, Jr., J. Appl. Phys. **30**, 625 (1959).
[5] A. Aharoni and S. Shtrikman, Phys. Rev. **109**, 1522 (1958).

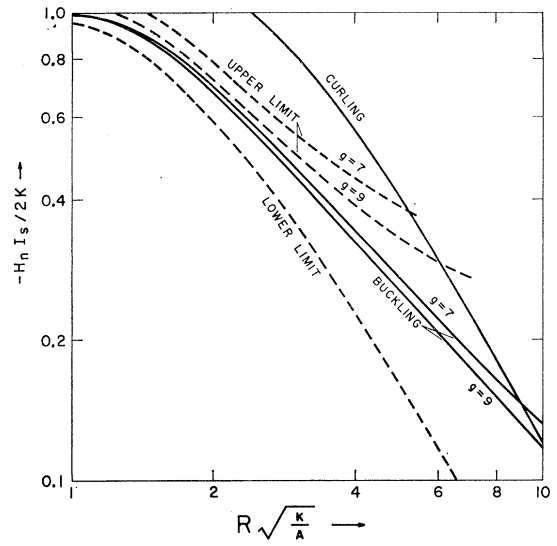

FIG. 1. The theoretical internal nucleation field H_n for an infinite ferromagnetic crystal which has a defective region in the form of a cyclinder of radius R in which the magnetocrystalline anisotropy vanishes. The nucleation field is plotted in terms of $2K/I_s$ which is the nucleation field for a perfect material ($R=0$). The upper and lower limits refer to the buckling mode ($m=1$), which depends on the additional parameter $g=K/\pi I_s^2$.

The other one need not be computed, since when the self-magnetostatic energy is neglected, this other determinant reduces to (12), so that its nucleation field cannot be less negative than the one given by (12). One can therefore conclude that for $m=0$ only A_φ is nonzero, and this is the curling mode.[5] Moreover, it is readily seen that the least negative eigenvalues of (12) are obtained for $k=0$, so that there is no z dependence. Using tabulated Bessel functions, values of μ_1 and ν_1 were chosen to satisfy (12), from which the relation between H, g, and S was computed using (11). Fortunately the relation between these three parameters can be given in terms of 2, namely, the reduced nucleation field as a function of reduced radius, containing the anisotropy K. The results of the computations in this case are plotted in Fig. 1, labeled "curling."

For $m\geq 2$ detailed calculation can again be avoided by neglecting the self-magnetostatic energy. The determinant D_m then breaks down into product of 2 determinants, yielding

$$J_{m-1}(i\mu_1)/\mu_1 J_m(i\mu_1)=H_{m-1}^{(1)}(i\nu_1)/\nu_1 H_m^{(1)}(i\nu_1), \quad (13a)$$

or

$$J_{m+1}(i\mu_1)/\mu_1 J_m(i\mu_1)=H_{m-1}^{(1)}(i\nu_1)/\nu_1 H_m^{(1)}(i\nu_1). \quad (13b)$$

Numerical computations showed that in each of the branches of (13) the nucleation field becomes more negative with increasing m, for $m\geq 1$, and that for $m\geq 2$ both branches give more negative nucleation field than the curling mode. One can, therefore, ignore all the $m\geq 2$ values and study only $m=1$. In this case, (13a) gives values of the nucleation field numerically smaller than in the curling case. These values, however, are only lower limits for the $m=1$ solution, since they are obtained by neglecting the (positive) self-magnetostatic energy, so that the actual nucleation should be more negative. The solution of (13a) for $m=1$ is plotted dashed in Fig. 1, labeled "lower limit." Again, since $k=0$, the nucleation field can be given as a functional dependence of 2 variables made from the three physical parameters of radius, anisotropy, and nucleation field.

Actually the solution of $D=0$ involves all three parameters. Moreover, there is the extra parameter k, which should be chosen so that H is the least negative, i.e., H should be minimized with respect to k. Now by differentiating $D=0$ and imposing $dH/dk=0$, one obtains $\partial D/\partial k=0$. One has, therefore, to solve the set of equations,

$$D_1=0, \quad \partial D_1/\partial k=0, \quad (14)$$

to obtain the nucleation field H_n as a function of the radius R and the anisotropy K.

The equation being very complicated, an enormous amount of numerical computations can be saved if one knows beforehand approximately where the solutions should be looked for. This calls for an upper limit, to be used together with the lower limit of Fig. 1.

Since the solution is minimized with respect to k, any choice of one particular value for k will give nucleation field which cannot be numerically smaller than the minimum. The most convenient choice is $k=0$. Results for $m=1$ and $k=0$ are plotted dashed in Fig. 1 labeled "upper limit." These are plotted for 2 values of g defined in (6) corresponding to two typical hard materials, namely $g=7$ which is the experimental value[6] for $BaFe_{12}O_{19}$ and $g=9$ which is the experimental value[7] for MnBi. These upper limits turn out to be close enough to the lower limit for practical purposes, so that solutions of (14), using the actual 6-order determinant, could be computed. The final optimal results for $m=1$ and the above-mentioned two values of g, are plotted in Fig. 1, labeled "buckling."

3. FINITE WIDTH

For finite q one should consider (4) and the additional boundary conditions involved by the continuity of u and its derivative at $p=\pm q$. Generally speaking this does not allow the separation of variables (8), and one has to use sums of the solutions of the previous section. However, one can expand the solution in terms of the eigenfunctions of the previous problem. For each mode discussed there, one can always write a unique solution of (4), using the values of u on $p=\pm q$ as boundary values for (4). The derivative of u on $p=\pm q$ will generally not be continuous on $p=\pm q$ for each of these functions, but this is equivalent to introducing surface charges, which cancel only for the sum of the functions.

[6] J. S. Smit and H. P. J. Wijn, *Ferrites* (John Wiley & Sons, Inc., New York, 1959), p. 204.

[7] R. M. Bozorth, *Ferromagnetism* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1951), p. 575.

According to (10b) these surface charges are proportional to modified Bessel functions of the third kind, which decay very rapidly in a distance of the order of the defective-region width. If the slab width is large compared to the defective-region width, these surface charges should thus be negligible for nucleation calculations with respect to the volume charges, involved in the calculation of each mode (no surface charges are involved there). This interaction of the modes is therefore evidently a negligible effect, and the lowest eigenvalues of the separate modes should be a good approximation for the lowest eigenvalues of the rigorous problem. One can, therefore, approximately ignore (4) and use the solutions (9) and (10) inside the material. There is just the extra condition (7), which yields, when substituted in (8),

$$k=n\pi/q, \quad p_0=0, \tag{15a}$$

or

$$k=(n+\tfrac{1}{2})\pi/q, \quad p_0=\pi/2, \tag{15b}$$

where n is an integer.

For the curling mode, as well as for the lower limits of $m\geq 1$ and for the upper limit of $m=1$ in Fig. 1, the value $k=0$ was obtained, so that one can use $n=0$ in (15a) and no change need be made in the previous results of Fig. 1. For the buckling mode, on the other hand, k takes an optimum value, which might not be an allowed value according to (15). The buckling graphs should therefore in principle lie higher than in Fig. 1. However, the computations of Fig. 1 showed that the minimum with respect to k is rather flat and H does not change appreciably by taking values of k around the optimal value. On the other hand, one is normally interested in slab width, which is large compared to the radius of dislocation, i.e., $q\gg 1$ according to (6). In this case (15) allows values of k to be rather close together, which lets one approach rather closely the optimal value of k. One can therefore consider Fig. 1 as giving as good an approximation as the whole rather crude model for a dislocation deserves. However, there is a large difference between these results for a slab and those obtained[2] for a material infinite in all directions, in that the field H_n in Fig. 1 is the *internal* field and to get it one should subtract the demagnetizing field of $4\pi I_s$ from the externally applied field. This can therefore yield a *positive* external nucleation field in hard materials, as will be discussed in the next section.

4. DISCUSSION

Kooy and Enz[8] have observed that domains in single crystals of $BaFe_{12}O_{19}$, in the form of thin platelets, nucleate at definite points in the crystals. The externally applied field at nucleation ranged from +500 to +2000 Oe, depending on the nucleation point. Taking[6] for $BaFe_{12}O_{19}$, $I_s=380$ G, and $K=3.3\times 10^6$ erg/cm^3, the observed external field of +500 corresponds to an *internal* field of about −4300 G, whereas for a perfect crystal, the theory predicts an internal nucleation field of $-2K/I_s$, i.e., about −17 000 G. According to Fig. 1, taking $g=7$ for $BaFe_{12}O_{19}$, the necessary factor of 4 reduction in $2K/I_s$ is realized for a defective region the reduced radius of which is about 6, i.e., 350 Å if[9] $A=10^{-6}$ erg/cm. This is not an unreasonable dimension for a dislocation. A reasonable radius for the imperfection can thus bring the theory to agreement with experiment for $BaFe_{12}O_{19}$ crystals.

The model of $K=0$ in a certain region is certainly a crude approximation for the change of anisotropy due to magnetostriction and the mechanical stresses in a dislocation. Moreover, nucleation of domains is probably due to stacking faults. One cannot, therefore, expect very rigorous results from it. However, the fact that the size of the imperfect region comes out about right is very encouraging. It therefore seems pretty well established that for hard materials (i.e., those in which $K\gg I_s^2$) the solution of the Brown paradox[3] lies mainly in the presence of dislocations. This would definitely not resolve the paradox for soft materials, i.e., materials in which $K\ll I_s^2$, as is clear from Fig. 1. Another important conclusion[3] from these results is that crystalline imperfections cannot affect much the nucleation field of *elongated* particles of hard materials, and this calls for a direct experimental test.

In the present treatment the nucleation problem reduced to the curling and buckling modes, the buckling being important for small radii, while the curling becomes easier for larger radii. In this respect the results are similar to those obtained for fine perfect particles.[2] However, the changeover from buckling to curling is at a larger radius than in the case of fine particles, while small radii are more practically realizable for imperfections than for particles. It therefore turns out that the most important mode, when imperfections are present, is the buckling, rather than the curling. The changes over from buckling to curling, according to Fig. 1, using the above mentioned experimental values for the parameters involved, is at a radius of about 300 Å for MnBi and 550 Å for $BaFe_{12}O_{19}$.

[8] C. Kooy and U. Enz, Philips Research Repts. **15**, 7 (1960).

[9] E. H. Frei, S. Shtrikman, and D. Treves, Phys. Rev. **106**, 446 (1957).