Magnetic Field Dependence of the Energy Gap in Superconductors*

YOICHIRO NAMBU
The Enrico Fermi Institute for Nuclear Studies, and The Department of Physics, University of Chicago, Chicago, Illinois

AND

SAN FU TUAN†
Department of Physics, Brown University, Providence, Rhode Island
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We calculate the reduction of the energy gap at zero temperature for a bulk superconductor in the presence of a static external magnetic field, to the second order in the field strength. Simple formulas are obtained for the long ($\lambda \gg \xi_0$) and short ($\lambda \ll \xi_0$) wavelength limits, where ξ_0 is the coherence length.

We use the general gauge-invariant formulation of the Bardeen-Cooper-Schrieffer (BCS)-Bogoliubov theory developed by one of the authors, but the result is also shown to agree with that of the BCS variational procedure applied in the presence of the field. The gauge invariance is maintained by virtue of the collective excitations as in the Meissner effect. The mathematical proof of gauge invariance is carried out in a completely general manner using Ward identities.

1. INTRODUCTION

We shall be concerned in this paper with the theoretical derivation of the magnetic field dependence of the energy gap in a superconductor at absolute zero temperature.

Recently, Douglass[1] has performed experiments on superconducting films which show a characteristic dependence of the energy gap on the magnetic field. He has been able to account for the results by means of the Ginzburg-Landau[2] equations which, according to Gor'kov,[3] follow from the Bardeen-Cooper-Schrieffer (BCS) theory[4] and describe just such an effect. However, the validity of Gor'kov's derivation is restricted to the London limit which occurs near the critical temperature. The calculation at zero temperature, which belongs to the Pippard case, has been done by Gupta and Mathur[5] using the theory of Wentzel.[6] Our calculation will also be performed at zero temperature, but in the framework of the gauge-invariant theory developed by one of the authors.[7] Our result will show that, to the second order in the field strength,

$$\delta(\phi^2) \sim -e^2 v_F^2 (\xi_0 q)^2 |\mathbf{A}(\mathbf{q})|^2, \quad (q\xi_0 = qv_F/\pi\phi \ll 1)$$

where ϕ is the gap, v_F is the Fermi velocity, ξ_0 is the coherence length, and $\mathbf{A}(\mathbf{q})$ is the Fourier component of the vector potential present in the medium.[8] The same result is obtained by a simple variational procedure similar to the original BCS work. But the central problem in this kind of derivation is the proof of gauge invariance. The result of Gupta and Mathur, though gauge invariant, gives $\delta(\phi^2) \sim -e^2 v_F^2 |\mathbf{A}(\mathbf{q})|^2$.

In Secs. 2 and 3 we shall present the general mathematical formulation and the actual derivation of the results for both small-q ($\ll \xi_0^{-1}$) and large-q ($\gtrsim \xi_0^{-1}$) regions.

The proof of gauge invariance of our procedure is carried out on a completely general basis in Sec. 4, by means of the Ward identities, whereas the equivalence of our result with the BCS variational procedure in the present problem is demonstrated in Sec. 5.

2. GENERAL FORMULATION

In a previous paper[7] the BCS-Bogoliubov theory was formulated on a general basis using the language and techniques of field theory. According to this, the problem reduces to finding the self-energy of a quasi-particle in the generalized Hartree-Fock approximation. We will briefly recapitulate the main points.

With the two-component notation combining the up-spin and down-spin electron wave functions,

$$\Psi(x) = \begin{pmatrix} \psi_\uparrow(x) \\ \psi_\downarrow^\dagger(x) \end{pmatrix} \quad \text{or} \quad \Psi(p) = \begin{pmatrix} \psi_\uparrow(p) \\ \psi_\downarrow^\dagger(-p) \end{pmatrix}, \quad (2.1)$$

the Hartree-Fock self-consistent equation for the self-energy Σ takes the form

$$\Sigma(x,y) = -\tau_3 G(x,y) \tau_3 V(x,y). \quad (2.2)$$

Here x and y in general refer to space-time coordinates; $G(x,y)$ is the Green's function for the quasi-particle satisfying

$$\begin{aligned} [i(\partial/\partial t) - H_0 - \Sigma] G(x,y) &= i\delta^4(x-y), \\ H_0 = \tau_3[(p^2/2m) - \mu] &= \tau_3 \epsilon_p, \end{aligned} \quad (2.3)$$

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† Present address: Purdue University, Lafayette, Indiana.

[1] D. H. Douglass, Jr., Phys. Rev. Letters **6**, 346 (1961); *ibid.* **7**, 14 (1961); Phys. Rev. **124**, 735 (1961).

[2] V. L. Ginzburg and L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **20**, 1064 (1950).

[3] L. P. Gor'kov, J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1918 (1959) [translation: Soviet Phys.—JETP **9**, 1364 (1959)].

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[5] K. K. Gupta and V. S. Mathur, Phys. Rev. **121**, 107 (1961).

[6] G. Wentzel, Phys. Rev. **111**, 1488 (1958); Phys. Rev. Letters **2**, 33 (1959). See also K. K. Gupta and V. S. Mathur, Phys. Rev. **115**, 75 (1959).

[7] Y. Nambu, Phys. Rev. **117**, 648 (1960), hereafter denoted I for reference purposes.

[8] Throughout the paper, natural units $\hbar = c = 1$ are taken.

and $-V(x,y)$ is the interaction potential which is usually assumed to be effective only for states near the Fermi surface, $p\sim p_F$; μ is the chemical potential, being equal to $p_F^2/2m$ for practical purposes. Equation (2.2) has a solution which contains a term $\tau_1\phi$, where $\phi(p)\approx$const is the energy gap parameter. We get

$$\phi=\phi\int\frac{d^3l}{2E_l}V(p-l)\approx\phi\rho\sinh^{-1}\frac{\omega}{\phi};\qquad \rho=\bar{V}N,\quad N=mp_F/2\pi^2,\quad (\bar{V}=\langle V\rangle_{\rm av}),\tag{2.4}$$

where ω is the Debye cutoff frequency. For small $\rho\ll 1$, ϕ becomes

$$\phi\approx 2\omega e^{-1/\rho}.\tag{2.5}$$

Now let us assume that a magnetic field represented by a vector potential $\mathbf{A}(x)$ is present in the medium. $\mathbf{A}(x)$ is the sum of the external $\mathbf{A}^{\rm ex}$ and the induced field, and should be zero in the case of infinite medium with uniform external field. The relation between the Fourier components of $\mathbf{A}$ and $\mathbf{A}^{\rm ex}$ is given by

$$A_i(\mathbf{q})=[1-K(q)/q^2]^{-1}A_i^{\rm ex}(\mathbf{q}),\tag{2.6}$$

where $K(q)$ is the kernel for the induced current

$$j_i(\mathbf{q})=K(q)A_i(\mathbf{q}).\tag{2.7}$$

Usually the kernel K is calculated with the solution of the free superconductive state, and is given by

$$-K(q)=(e^2/m)n+O(q^2)\equiv(1/\Lambda)+O(q^2),\qquad n=p_F^3/3\pi^2.\tag{2.8}$$

This will lead to the Meissner effect, but we do not yet have the field dependence of the energy gap.

In order to obtain the field dependence of the gap, it is necessary to set up the self-energy equation in the presence of the field. Thus, we will modify Eqs. (2.2), (2.3) as

$$\Sigma^{(A)}(x,y)=-\tau_3G^{(A)}(x,y)\tau_3V(x,y),\tag{2.9}$$

$$\begin{aligned}&[i(\partial/\partial t)-H_0^{(A)}-\Sigma^{(A)}]G^{(A)}(x,y)=i\delta^4(x-y),\\ &H_0^{(A)}=\tau_3\{[(\mathbf{p}-e\tau_3\mathbf{A})^2/2m]-\mu^{(A)}\}\\ &\qquad=\tau_3\epsilon_p-(e/m)\mathbf{p}\cdot\mathbf{A}+(e^2/2m)\tau_3A^2-\delta\mu\tau_3.\end{aligned}\tag{2.9'}$$

The new solution can then be determined by expanding everything in powers of $\mathbf{A}$. Let us define

$$\begin{aligned}\Sigma^{(A)}(x,y)&=\Sigma(x-y)+\int\left[\frac{\delta\Sigma^{(A)}(x,y)}{\delta A_i(z)}\right]_{A=0}A_i(z)dz+\tfrac{1}{2}\int\int\left[\frac{\delta^2\Sigma^{(A)}(x,y)}{\delta A_i(z)\delta A_j(z')}\right]_{A=0}A_i(z)A_j(z')dzdz'+\cdots,\\ &[\delta\Sigma^{(A)}(x,y)/\delta A_i(z)]_{A=0}\equiv-e\Lambda_i(x,y;z),\quad [\delta^2\Sigma^{(A)}/\delta A_i\delta A_j]_{A=0}\equiv e^2\Lambda_{ij}(x,y;z,z').\end{aligned}\tag{2.10}$$

From Eq. (2.9') we get accordingly

$$\begin{aligned}G^{(A)}(x,y)&=G(x-y)+\int\left[\frac{\delta G^{(A)}(x,y)}{\delta A_i(z)}\right]_{A=0}A_i(z)dz+\tfrac{1}{2}\int\int\left[\frac{\delta^2G^{(A)}(x,y)}{\delta A_i(z)\delta A_j(z')}\right]_{A=0}A_i(z)A_j(z')dzdz',\\ &[\delta G^{(A)}(x,y)/\delta A_i(z)]_{A=0}=eiG\Gamma_iG,\end{aligned}\tag{2.10'}$$

$$\Gamma_i(x,y;z)=i\delta G^{-1}/\delta A_i\equiv e(\gamma_i+\Lambda_i)=-(ei/m)[(\partial/\partial x_i-\partial/\partial y_i)\delta^3(x-y)]\delta^3(x-z)+e\Lambda_i(x,y;z),$$

$$\begin{aligned}[\delta^2G^{(A)}(x,y)/\delta A_i(z)\delta A_j(z')]_{A=0}&\equiv-e^2L_{ij}(x,y;z,z')=\left[ie\frac{\delta}{\delta A_j(z')}(G\Gamma_iG)\right]_{A=0}\\ &=\left[ie\left(\frac{\delta G}{\delta A_j}\Gamma_iG+G\Gamma_i\frac{\delta G}{\delta A_j}+G\frac{\delta\Gamma_i}{\delta A_j}G\right)\right]_{A=0}=-e^2(G\Gamma_jG\Gamma_iG+G\Gamma_iG\Gamma_jG-iG\Gamma_{ij}G),\end{aligned}\tag{2.11}$$

$$e^2\Gamma_{ij}(x,y;z,z')=e[\delta\Gamma_i(x,y;z)/\delta A_j(z')]_{A=0}=-(e^2/m)\delta_{ij}\delta^3(x-y)\delta^3(x-z)\delta^3(z-z')+e^2\Lambda_{ij}(x,y;z,z').$$

Here the obvious shorthand writing such as

$$G\Gamma_iG=\int\int G(x,x')\Gamma_i(x',y';z)G(y',y)d^4x'd^4y'$$

has been used. Substituting Eqs. (2.10) and (2.11) into Eq. (2.9), we obtain the relations

$$\Sigma(x,y)=-\tau_3G(x,y)\tau_3V(x,y),\tag{2.12a}$$

$$\Lambda_i(x,y;z)=i\int\int\tau_3G(x,x')\Gamma_i(x',y';z)G(y',y)\tau_3V(x,y)d^4x'd^4y',\tag{2.12b}$$

$$\Lambda_{ij}(x,y;z,z')=-\tau_3L_{ij}(x,y;z,z')\tau_3V(x,y)=-\tau_3(G\Gamma_jG\Gamma_iG+G\Gamma_iG\Gamma_jG+G\Gamma_{ij}G)\tau_3V.\tag{2.12c}$$

These will determine Σ, Λ_i, $\Lambda_{ij}\cdots$ successively. In general, they are operators containing 1, τ_3, τ_1, τ_2 in the two-component notation. We are, however, interested in that part of Λ_{ij} which is proportional to τ_1, since this gives the correction to the energy gap term in Σ which is also proportional to τ_1.

In the following we will assume that $\Sigma=\tau_1\phi$, $\Gamma_i=\gamma_i$, and with this calculate Λ_{ij} using Eq. (2.12c). The reason is that other terms are either renormalization terms (τ_3 in Σ) or are important only for unphysical longitudinal potentials (τ_2 in Γ_i), so that they will not cause any physically significant changes in our results. For the discussion of gauge invariance of the results, of course, we have to examine those terms which are neglected. We will do this in a later section.

3. CALCULATIONS

The compensation equation (2.9) written in momentum space depends on the initial and final momenta of the quasi-particle. For practical purposes, however, we restrict ourselves to the diagonal elements ($p=p'$). We get

$$\Sigma^{(A)}(p)=\frac{-1}{(2\pi)^4}\int\tau_3 G^{(A)}(\mathbf{l},l_0)\tau_3 V(\mathbf{p}-\mathbf{l},\,p_0-l_0)d^4l. \quad (3.1)$$

Here

$$\begin{aligned}G^{(A)}(\mathbf{l},l_0)&=i/[l_0-H^{(A)}(\mathbf{l})]\\&=i\bigg[\frac{1}{l_0-H(\mathbf{l})}-\frac{1}{l_0-H(\mathbf{l})}\frac{e\mathbf{l}\cdot\mathbf{A}}{m}\frac{1}{l_0-H(\mathbf{l})}\\&\quad+\frac{1}{l_0-H(\mathbf{l})}\frac{e\mathbf{l}\cdot\mathbf{A}}{m}\frac{1}{l_0-H(\mathbf{l})}\frac{e\mathbf{l}\cdot\mathbf{A}}{m}\frac{1}{l_0-H(\mathbf{l})}\\&\quad+\frac{1}{l_0-H(\mathbf{l})}\tau_3\frac{e^2|\mathbf{A}|^2}{2m}\frac{1}{l_0-H(\mathbf{l})}\\&\quad+\frac{1}{l_0-H(\mathbf{l})}\Sigma^{(2)}\frac{1}{l_0-H(\mathbf{l})}\bigg]+O(A^3),\end{aligned}$$

$$H(\mathbf{l})=H_0+\Sigma=\epsilon\tau_3+\tau_1\phi.$$

The zeroth order term $\Sigma^{(0)}(p)$ is given by

$$\begin{aligned}\Sigma^{(0)}(p)&=\frac{-i}{(2\pi)^4}\int\tau_3\frac{d^4l}{l_0-H(\mathbf{l})}\tau_3V(p-l)\\&=-\frac{1}{2(2\pi)^3}\tau_3\int\frac{\epsilon(\mathbf{l})\bar{V}d^3l}{E(\mathbf{l})}+\frac{1}{2(2\pi)^3}\tau_1\int\frac{\phi}{E(\mathbf{l})}\bar{V}d^3l, \quad (3.2)\end{aligned}$$

where the effective potential is restricted to near the Fermi surface. The first term on the right-hand side makes little contribution, and the second yields the energy gap equation

$$\frac{\tau_1}{2(2\pi)^3}\int\frac{\phi\bar{V}}{E(\mathbf{l})}d^3l=\frac{\tau_1}{2}\int_{-\omega}^{\omega}\frac{\phi\bar{V}N(\mathbf{l})d\epsilon(\mathbf{l})}{[\epsilon^2(\mathbf{l})+\phi^2]^{1/2}}=\tau_1\phi.$$

Hence

$$\Sigma^{(0)}(p)=\tau_1\phi. \quad (3.3)$$

The first-order term $\Sigma^{(1)}$ is the vertex correction which has been investigated in I (cf. reference 7) and does not concern us now. The second-order contribution $\Sigma^{(2)}$ contains three terms,

$$\Sigma^{(2)}=\Sigma_1^{(2)}+\Sigma_2^{(2)}+\Sigma_3^{(2)}.$$

Here $\Sigma_1^{(2)}$ is the contribution from the diagrams of Figs. 1(a) and 1(b), while $\Sigma_2^{(2)}$ and $\Sigma_3^{(2)}$ correspond to Figs. 1(c) and 1(d), respectively. We will consider one particular momentum $\mathbf{q}$ at a time. We have

$$\begin{aligned}\Sigma_1^{(2)}(p)&=\frac{-i}{(2\pi)^4}\int\tau_3\frac{V}{l_0-H(\mathbf{l})}\frac{e\mathbf{l}\cdot\mathbf{A}(-\mathbf{q})}{m}\\&\quad\times\frac{1}{l_0-H(\mathbf{l}+\mathbf{q})}\frac{e\mathbf{l}\cdot\mathbf{A}(\mathbf{q})}{m}\frac{1}{l_0-H(\mathbf{l})}\tau_3d^4l\\&\quad+\frac{-i}{(2\pi)^4}\int\tau_3\frac{V}{l_0-H(\mathbf{l})}\frac{e\mathbf{l}\cdot\mathbf{A}(\mathbf{q})}{m}\frac{1}{l_0-H(\mathbf{l}-\mathbf{q})}\\&\quad\times\frac{e\mathbf{l}\cdot\mathbf{A}(-\mathbf{q})}{m}\frac{1}{l_0-H(\mathbf{l})}\tau_3d^4l\equiv M_1+M_2. \quad (3.4)\end{aligned}$$

M_2 is obtained from M_1 by substitution $\mathbf{q}\rightarrow-\mathbf{q}$. M_1 may be written after integrating out l_0 as

$$M_1=-\frac{1}{2(2\pi)^3}\int\frac{e^2}{m^2}\mathbf{l}\cdot\mathbf{A}(-\mathbf{q})\mathbf{l}\cdot\mathbf{A}(\mathbf{q})\bar{V}Xd^3l, \quad (3.5)$$

with

$$\begin{aligned}X&=-A\frac{1}{2E(\mathbf{l})[E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]^2}\\&\quad+B\frac{2E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})}{2E^3(\mathbf{l})E(\mathbf{l}+\mathbf{q})[E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]^2},\\A&=[2\epsilon(\mathbf{l})+\epsilon(\mathbf{l}+\mathbf{q})]\tau_3-3\phi\tau_1,\\B&=[\epsilon(\mathbf{l})\tau_3-\tau_1\phi][\epsilon(\mathbf{l}+\mathbf{q})\tau_3-\tau_1\phi][\epsilon(\mathbf{l})\tau_3-\tau_1\phi],\\&=\{\epsilon^2(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{q})+\phi^2[2\epsilon(\mathbf{l})-\epsilon(\mathbf{l}+\mathbf{q})]\}\tau_3\\&\quad+\phi[\epsilon^2(\mathbf{l})-2\epsilon(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{q})-\phi^2]\tau_1.\end{aligned} \quad (3.6)$$

For $\mathbf{q}=0$, we have $X=0$.

To calculate the London limit $q\ll\xi_0^{-1}$, we may conveniently expand $\epsilon(\mathbf{l}+\mathbf{q})$ in powers of $\mathbf{q}$, retaining for the integrand X terms of order q^2. Making use of the reality condition $A(-\mathbf{q})=A^*(\mathbf{q})$, we get for the τ_1 part of M_1

$$\begin{aligned}(M_1)_{\tau_1}&=-\frac{\tau_1}{2(2\pi)^3}\int\frac{e^2}{m^2}\bar{V}|\mathbf{l}\cdot\mathbf{A}(\mathbf{q})|^2d^3l\bigg[-\frac{\phi\epsilon(\mathbf{l})}{4E^5(\mathbf{l})}\frac{q^2}{m^2}\\&\quad+\frac{\phi}{8E^5(\mathbf{l})}\frac{(\mathbf{q}\cdot\mathbf{l})^2}{m^2}\frac{\phi^2}{E^2}+\frac{3}{4}\frac{\phi\epsilon^2(\mathbf{l})}{E^7(\mathbf{l})}\left(\frac{\mathbf{q}\cdot\mathbf{l}}{m}\right)^2\bigg]\\&=-(1/90)(\tau_1/\phi)(e^2/m^2)\\&\quad\times\rho q^2|\mathbf{A}(\mathbf{q})|^2(p_F^4/m^2\phi^2). \quad (3.7)\end{aligned}$$

We have restricted the integration to the range $-\omega<\epsilon(\mathbf{l})<\omega$, and neglected terms of the order $(\phi/\omega)^2$. Obviously M_1 is symmetric under $\mathbf{q}\to-\mathbf{q}$, hence $(M_2)_{\tau_1}=(M_1)_{\tau_1}$, and

$$[\Sigma_1^{(2)}(p)]_{\tau_1}=\delta\phi\tau_1=-\frac{1}{45}\frac{\tau_1}{\phi}\left(\frac{e^2}{m^2}\right)\rho q^2|\mathbf{A}(\mathbf{q})|^2\frac{p_F^4}{m^2\phi^2}. \quad (3.8)$$

The next term $(\Sigma_2^{(2)})_{\tau_1}$ is readily shown to vanish independently of the magnitude of $\mathbf{q}$. In fact, for the more general situation $\mathbf{p}\neq\mathbf{p}'$ ($\mathbf{q}\neq\mathbf{q}'$), we have

$$\Sigma_2^{(2)}(\mathbf{p}',\mathbf{p})=\frac{-i}{(2\pi)^4}\int\tau_3\frac{1}{l_0-H(\mathbf{l}+\mathbf{\Delta})}\tau_3\frac{e^2}{2m}\times\mathbf{A}(\mathbf{q})\cdot\mathbf{A}(-\mathbf{q}')\frac{1}{l_0-H(\mathbf{l})}\tau_3V(p-l)d^4l$$

$$=\frac{-i}{(2\pi)^4}\frac{e^2}{2m}\mathbf{A}(\mathbf{q})\cdot\mathbf{A}(-\mathbf{q}')\times\int\frac{[l_0^2+\epsilon(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{\Delta})-\phi^2]\tau_3-\phi[\epsilon(\mathbf{l}+\mathbf{\Delta})+\epsilon(\mathbf{l})]\tau_1}{[l_0^2-E^2(\mathbf{l}+\mathbf{\Delta})][l_0^2-E^2(\mathbf{l})]}\times V(p-l)d^4l,$$

where

$$\mathbf{\Delta}=\mathbf{q}-\mathbf{q}'=\mathbf{p}'-\mathbf{p}.$$

Hence

$$[\Sigma_2^{(2)}(\mathbf{p}',\mathbf{p})]_{\tau_1}=\frac{i}{(2\pi)^4}\tau_1\frac{e^2}{2m}\frac{\phi}{2m}\mathbf{A}(\mathbf{q})\cdot\mathbf{A}(-\mathbf{q}')\times\int\frac{\mathbf{\Delta}^2+2(l^2-p_F^2)}{[l_0^2-E^2(\mathbf{l})]^2}V(p-l)d^4l$$

$$=-(e^2/8m^2)(\rho/\phi)(\mathbf{p}'-\mathbf{p})^2\mathbf{A}(\mathbf{q})\cdot\mathbf{A}(-\mathbf{q}')\tau_1, \quad (3.9)$$

which vanishes for $\mathbf{p}=\mathbf{p}'$.

The third term $\Sigma_3^{(2)}$ is the radiative correction to $\Sigma^{(2)}$ itself which must be added in order to obtain self-consistency to this order. Its contribution to the τ_1 part can be calculated from the BCS energy gap Eq. (3.2) as

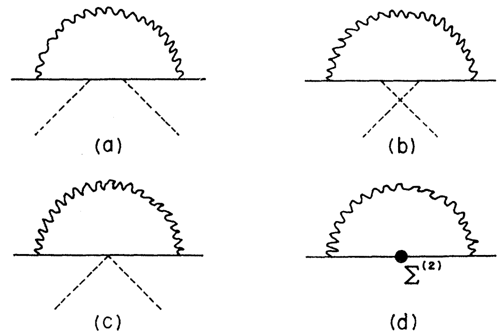

FIG. 1. Compensation equation diagrams to second order in the external electromagnetic field A_μ.

$$[\Sigma_2^{(2)}(\mathbf{p})]_{\tau_1}=\tau_1\delta\int\frac{\phi}{2E(\mathbf{l})}\bar{V}d^3l=\tau_1\delta\phi\left[\int\frac{N\bar{V}d\epsilon}{2E}-\int\frac{\phi^2}{2E^3}N\bar{V}d\epsilon\right]=\tau_1(1-\rho)\delta\phi. \quad (3.10)$$

Combining Eqs. (3.8) and (3.10), the compensation equation becomes

$$\delta\phi=(1-\rho)\delta\phi-(1/45\phi)(e^2/m^2)\rho q^2|\mathbf{A}(\mathbf{q})|^2(p_F^4/m^2\phi^2), \quad (3.11)$$

or

$$\delta\phi=-(1/45\phi)(e^2/m^2)q^2|\mathbf{A}(\mathbf{q})|^2(p_F^4/m^2\phi^2).$$

Introducing the coherence length $\xi_0=v_F/\pi\phi$ and summing over all different Fourier components $\mathbf{A}(\mathbf{q})$, we obtain

$$\delta(\phi^2)=-(\pi^2/45)e^2v_F^2\sum_{\mathbf{q}}(\xi_0\mathbf{q})^2|\mathbf{A}(\mathbf{q})|^2, \quad (3.12)$$

where $\mathbf{q}$ and $-\mathbf{q}$ are to be counted independently.

We now briefly comment on the τ_3 (kinetic energy) part of $\Sigma^{(2)}$. From Eqs. (3.4)–(3.6), we have

$$[\Sigma_1^{(2)}(\mathbf{p})]_{\tau_3}=-\frac{1}{(2\pi)^3}\int\frac{e^2}{m^2}\mathbf{l}\cdot\mathbf{A}(-\mathbf{q})\mathbf{l}\cdot\mathbf{A}(\mathbf{q})[X]_{\tau_3}\bar{V}d^3l;$$
$$[X]_{\tau_3}=\tau_3\{2E^2(\mathbf{l})[E(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{q})-E(\mathbf{l}+\mathbf{q})\epsilon(\mathbf{l})]+2\phi^2[\epsilon(\mathbf{l})-\epsilon(\mathbf{l}+\mathbf{q})][2E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]\}/2E^3(\mathbf{l})E(\mathbf{l}+\mathbf{q})[E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]^2. \quad (3.13)$$

Figures 1(c) and 1(d) give, respectively,

$$[\Sigma_2^{(2)}(\mathbf{p})]_{\tau_3}=\frac{-i}{(2\pi)^4}\frac{2e^2|\mathbf{A}(\mathbf{q})|^2\tau_3}{2m}\times\int\frac{[l_0^2+\epsilon^2(\mathbf{l})-\phi^2]V(p-l)d^4l}{[l_0^2-E^2(\mathbf{l})]^2}=(e^2/m)|\mathbf{A}(\mathbf{q})|^2\rho\tau_3, \quad (3.14)$$

$$[\Sigma_3^{(2)}(\mathbf{p})]_{\tau_3}=\tau_3\delta\epsilon(\mathbf{p})=\tau_3\delta\int\frac{\epsilon}{2E}N\bar{V}d\epsilon=\tau_3\left[\int\frac{\phi^2}{2E^3}N\bar{V}d\epsilon\right]\delta\epsilon=\tau_3\rho\delta\epsilon. \quad (3.15)$$

Combining Eqs. (3.13)–(3.15), we obtain

$$\tau_3\delta\epsilon(\mathbf{p})=\tau_3\rho\delta\epsilon(\mathbf{p})+\tau_3\frac{e^2}{m}|\mathbf{A}(\mathbf{q})|^2\rho+\Sigma_1^{(2)}(\mathbf{p}),$$

or

$$(1-\rho)\delta\epsilon(\mathbf{p})=\frac{e^2}{m}|\mathbf{A}(\mathbf{q})|^2\rho+[\Sigma_1^{(2)}(\mathbf{p})]_{\tau_3}/\tau_3. \quad (3.16)$$

As is seen from Eq. (3.13), $\Sigma_1^{(2)}(\mathbf{p})$ is independent of p near the Fermi surface, and so is $\delta\epsilon(\mathbf{p})$. This constant shift in ϵ is equivalent to a shift in the chemical potential. Since the particle number n has to be kept fixed, and this depends essentially on the chemical potential, we have to subtract away $\delta\epsilon=\delta\mu$ as the renormalization of the chemical potential.

It must be noted that the "Compton diagrams," which are second order in $\mathbf{A}$, do not contribute to the compensation Eq. (3.1), and hence to the energy gap $\phi(\mathbf{A})$. This is so because the self-energy Σ in the compensation equation is the proper self-energy in the terminology of quantum electrodynamics, whereas the Compton diagrams belong to the improper self-energy. The Compton diagrams are relevant if we are interested in the solution of the quasi-particle Schrödinger equation $H^{(A)}\Psi=E\Psi$. These diagrams then contribute to the second-order term $\delta E^{(2)}$, and will affect the p dependence of E, but not the gap itself. We also emphasize that the Compton diagrams are needed in the discussion of the over-all gauge invariance of the theory as will be shown in Sec. 4.

In Sec. 2 we observed that the field $\mathbf{A}$ is related to the external field $\mathbf{A}^{\text{ex}}$ via Eq. (2.6) which depends on the polarization kernel K. For later purposes we will calculate here $K(q)$ to the second order in q. Following I, K is given by

$$K_{ij}(q)=-(ne^2/m)\delta_{ij}+K_{ij}^{(2)}(q),$$

$$K_{ij}^{(2)}(q)=\frac{-ie^2}{(2\pi)^4}\int \mathrm{Tr}[\gamma_i(p-\tfrac{1}{2}q,\,p+\tfrac{1}{2}q)G(p+\tfrac{1}{2}q) \quad (3.17)$$
$$\times\Gamma_j(p+\tfrac{1}{2}q,\,p-\tfrac{1}{2}q)G(p-\tfrac{1}{2}q)]d^4p.$$

Γ_j may be replaced by γ_j for transversal waves, so that

$$K_{ij}^{(2)}(q)=\frac{-ie^2}{(2\pi)^4}\int \mathrm{Tr}\left[\frac{p_i}{m}G(p+\tfrac{1}{2}q)\frac{p_j}{m}G(p-\tfrac{1}{2}q)\right]d^4p$$
$$=\frac{-ie^2}{(2\pi)^4}i\pi\int\frac{p_ip_j}{m^2}(\mathbf{p}\cdot\mathbf{q})^2\frac{\phi^2}{m^2}\frac{d^3p}{2E^5(\mathbf{p})}. \quad (3.18)$$

Making use of the relation

$$\langle p_ip_jp_kp_m\rangle_{\text{av}}=\langle p^4\rangle_{\text{av}}(\delta_{ik}\delta_{jm}+\delta_{ij}\delta_{km}+\delta_{im}\delta_{kj})/15,$$

we have

$$K_{ij}^{(2)}(q)=(1/45)e^2N[q^2\delta_{ij}+2q_iq_j]p_F^4/m^4\phi^2,$$

or

$$K_{ij}(q)=-(ne^2/m)\delta_{ij}+(e^2/45)N(v_F^4/\phi^2)[q^2\delta_{ij}+2q_iq_j].$$

As was demonstrated in I, the effect of the collective excitations on Γ in Eq. (3.17) is to make the above form vanish for a longitudinal potential. Thus, the correct gauge-invariant result is

$$K_{ij}(q)=-(ne^2/m)(\delta_{ij}-q_iq_j/q^2)$$
$$+(e^2/45)N(v_F^4/\phi^2)[q^2\delta_{ij}-q_iq_j]$$
$$=-(ne^2/m)(\delta_{ij}-q_iq_j/q^2)[1-(\pi^2/30)(\xi_0q)^2]. \quad (3.19)$$

It is appropriate to discuss briefly the type of energy gap behavior expected for large q. For this purpose, we have to evaluate the integral (3.5) without expansion in $\mathbf{q}$. The τ_1 part of $\Sigma_1^{(2)}$ reads

$$[\Sigma_1^{(2)}(\mathbf{p})]_{\tau_1}$$
$$=-\frac{1}{2}\frac{1}{(2\pi)^3}\sum_{\pm\mathbf{q}}\int\frac{e^2}{m^2}\mathbf{l}\cdot\mathbf{A}(-\mathbf{q})\mathbf{l}\cdot\mathbf{A}(\mathbf{q})\bar{V}(X)_{\tau_1}d^3l,$$

$$(X)_{\tau_1}=\frac{3\phi\tau_1}{2E(\mathbf{l})[E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]^2} \quad (3.20)$$
$$+\frac{\phi\tau_1[\epsilon^2(\mathbf{l})-2\epsilon(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{q})-\phi^2][2E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]}{2E^3(\mathbf{l})E(\mathbf{l}+\mathbf{q})[E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})]^2}.$$

Although this expression is quite complicated, the integral can be expressed, upon summation over $\pm\mathbf{q}$ (and change of variable $\mathbf{l}-\mathbf{q}\to\mathbf{l}$ for the case $-\mathbf{q}$), in the simpler form

$$[\Sigma_1^{(2)}(\mathbf{p})]_{\tau_1}=-\frac{1}{4}\frac{1}{(2\pi)^3}\tau_1\frac{\partial}{\partial\phi}\int\frac{e^2}{m^2}\mathbf{l}\cdot\mathbf{A}(-\mathbf{q})\mathbf{l}\cdot\mathbf{A}(\mathbf{q})\bar{V}d^3l$$
$$\times\frac{1}{E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})}\left[1-\frac{\epsilon(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{q})+\phi^2}{E(\mathbf{l})E(\mathbf{l}+\mathbf{q})}\right]. \quad (3.21)$$

Before discussing this integral, we will next consider the polarization kernel $K(q)$ for large q. Without expanding Eq. (3.18) in q, we have (for the transverse part)

$$K_{ij}^{(2)}(q)=\frac{1}{(2\pi)^3}\frac{e^2}{m^2}\int l_il_j\frac{1}{E(\mathbf{l})+E(\mathbf{l}+\mathbf{q})}$$
$$\times\left[1-\frac{\epsilon(\mathbf{l})\epsilon(\mathbf{l}+\mathbf{q})+\phi^2}{E(\mathbf{l})E(\mathbf{l}+\mathbf{q})}\right]d^3l, \quad (3.22)$$

which depends essentially on the same integral that appeared in Eq. (3.21). This is not an accident, as will be shown in a more general way in Sec. 5, where we discuss the equivalence of our method to the variational procedure.

Equation (3.22) has been evaluated by Bardeen, Cooper, and Schrieffer.[4] Their result is

$$K_{ij}(q)=K_{ij}^{(1)}(q)+K_{ij}^{(2)}(q)=-\frac{ne^2}{m}\delta_{ij}+K_{ij}^{(2)}(q)$$
$$=-[\delta_{ij}-q_iq_j/q^2]\frac{3\pi^2}{4}\frac{ne^2\phi}{qmv_F}\left[1-\frac{16\phi}{\pi^2qv_F}\ln(\pi q\xi_0)\right]$$
$$+[K_{ij}(q)]_{\phi=0}. \quad (q\xi_0\gg1) \quad (3.23)$$

Making use of Eq. (3.23), the quantity $[\Sigma_1^{(2)}(\mathbf{p})]_{\tau_1}$ can be easily evaluated. We find

$$[\Sigma_1^{(2)}(p)]_{\tau_1}=-\tau_1\frac{1}{4}\left(\frac{\rho}{N}\right)\frac{3\pi^2}{4}\frac{ne^2}{qmv_F}\times\left[1-\frac{32\phi}{\pi^2 qv_F}\ln(\pi q\xi_0)+\frac{16\phi}{\pi^2 qv_F}\right]. \quad (3.24)$$

This, together with Eqs. (3.8) and (3.10), yields finally

$$\delta\phi=-\frac{\pi^2}{8}e^2v_F\sum_q\frac{1}{q}\left[1-\frac{16\phi}{\pi^2qv_F}\{2\ln(\pi q\xi_0)-1\}\right]|\mathbf{A}(\mathbf{q})|^2. \quad (3.25)$$

4. PROOF OF GAUGE INVARIANCE

The proof of gauge invariance of our theory is completely general as well as simple. Our procedure will be to show that (a) there exist Ward identities[9] as rigorous manifestations of the gauge invariance, and (b) our Hartree-Fock approximation satisfies these identities.

The (generalized) Ward identities are relations between successive terms of the expansion of the Green's function $G^{(A)}$ in the field A as was carried out in Sec. 2. We start from the gauge invariance condition

$$e^{ie\lambda(x)\tau_3}G(x,y)e^{-ie\lambda(y)\tau_3}\equiv e^{ie\lambda\tau_3}Ge^{-ie\lambda\tau_3}=G^{(A')},\quad A_\mu'=A_\mu+\partial_\mu\lambda. \quad (4.1)$$

By expanding both sides in λ after putting $A_\mu=0$, and using the results of Sec. 2,[10] we first obtain

$$\begin{aligned}ie[\lambda\tau_3,G]&=\int\frac{\delta G}{\delta A_\mu(z)}\partial_\mu\lambda(z)d^4z\\&=ie\int G\Gamma_\mu G\partial_\mu\lambda(z)d^4z\\&=-ie\int G\partial_\mu\Gamma_\mu G\lambda(z)d^4z.\end{aligned} \quad (4.2)$$

In the written-out form,

$$\tau_3\lambda(x)G(x,y)-G(x,y)\tau_3\lambda(y)=-\int\int\int G(x,x')\times\frac{\partial\Gamma_\mu}{\partial z_\mu}(x',y';z)G(y',y)\lambda(z)d^4x'd^4y'd^4z. \quad (4.2')$$

This is the equivalent of the Ward-Takahashi identity for the vertex function Γ_μ. Multiplying (4.2) by G^{-1} from the left and from the right, we obtain also

$$[G^{-1},\tau_3\lambda]=-\int\frac{\partial\Gamma_\mu}{\partial z_\mu}\lambda(z)d^4z. \quad (4.3)$$

This shows that the current is divergenceless when taken between real one-particle states for which we have $G^{-1}=0$.

The above relation has already been utilized in the previous paper I for the proof of gauge invariance of the Meissner effect. For the present problem we need an analogous relation for the two-photon vertex $\Gamma_{\mu\nu}$. This can be done by considering $\delta G/\delta A_\mu$ instead of G in Eqs. (4.2) and (4.3):

$$ie\left[\lambda\tau_3,\frac{\delta G}{\delta A_\mu(z)}\right]=\int\frac{\delta}{\delta A_\nu(z')}\left[\frac{\delta G}{\delta A_\mu(z)}\right]\partial_\nu\lambda(z')d^4z',$$

or

$$[\lambda\tau_3,G\Gamma_\mu G]=-\int\partial_\nu'L_{\mu\nu}\lambda'd^4z'. \quad (4.4)$$

Now

$$\begin{aligned}\int\partial_\nu'L_{\mu\nu}\lambda'dz'&=\int\{G\partial_\nu'\Gamma_\nu G\Gamma_\mu G+G\Gamma_\mu G\partial_\nu'\Gamma_\nu G+G\partial_\nu'\Gamma_{\mu\nu}G\}\lambda'dz'\\&=-[\lambda\tau_3,G]\Gamma_\mu G-G\Gamma_\mu[\lambda\tau_3,G]+i\int G\partial_\nu'\Gamma_{\mu\nu}G\lambda'd^4z'\\&=-[\lambda\tau_3,G\Gamma_\mu G]+G[\lambda\tau_3,\Gamma_\mu]G+i\int G\partial_\nu'\Gamma_{\mu\nu}G\lambda'd^4z',\end{aligned}$$

where Eq. (4.2) is utilized. Comparing this with the left-hand side of Eq. (4.4), we get

$$[\lambda\tau_3,\Gamma_\mu]=-i\int\partial_\nu'\Gamma_{\mu\nu}\lambda'd^4z', \quad (4.5)$$

which is the desired relation for $\Gamma_{\mu\nu}$. The second-order scattering amplitude of the quasi-particle by $A(q)$ is $G^{-1}L_{\mu\nu}G^{-1}A_\mu A_\nu'$. Replacing, say, A_ν' by a longitudinal potential $\partial_\nu'\lambda'$ results in a zero matrix element on the mass shell in view of Eq. (4.4), as is physically required. This is achieved by the cancellation of the three terms making up $L_{\mu\nu}$.

Our next task is to show the consistency of the Hartree-Fock solution with gauge invariance. We expect the self-energy equation (2.9) to satisfy the gauge condition. In other words, a solution $G^{(A)}$ (or $\Sigma^{(A)}$) of Eq. (2.9) must satisfy Eq. (4.1), and consequently solutions Γ_μ, $\Gamma_{\mu\nu}$ of Eq. (2.12) must satisfy the corresponding Ward identities (4.3) and (4.5). This can easily be seen to be the case by taking the latter identities as an ansatz for $\partial_\mu\Gamma_\mu$, $\partial_\nu'\Gamma_{\mu\nu}$. In fact, assume Eqs. (4.2), (4.3) and put this into the right-hand side

[9] J. C. Ward, Phys. Rev. **78**, 182 (1950); Proc. Phys. Soc. (London) **A64**, 54 (1951). Y. Takahashi, Nuovo cimento **6**, 370 (1957).

[10] The formulation in this section will be written in four-vector notation; $\mu=1, 2, 3, 4$. It is evident that Eqs. (2.10)–(2.12) of Sec. 2 can be easily recast in the four-verctor form.

of the integral equation for $\partial_\mu \Gamma_\mu$:

$$\partial_\mu \Lambda_\mu = i(\tau_3 G \partial_\mu \Gamma_\mu G \tau_3) V. \tag{4.6}$$

We get

$$\int (\tau_3 G \partial_\mu \Gamma_\mu G \tau_3) V \lambda d^4 z = -(\tau_3 [\tau_3 \lambda, G] \tau_3) V$$
$$= -[\tau_3 \lambda, (\tau_3 G \tau_3 V)] = [\tau_3 \lambda, \Sigma].$$

But this is exactly the relation (4.3) assumed for the right-hand side of Eq. (4.6). Although we took an integral of Eq. (4.6) with $\lambda(z)$, it is equivalent to the unintegrated form since $\lambda(z)$ is arbitrary. Namely, by taking $\lambda(z') = \delta(z'-z)$, we find that

$$(\partial \Gamma_\mu / \partial z_\mu)(x,y;z)$$
$$= i[G^{-1}(x,y)\tau_3 \delta^4(y-z) - \tau_3 \delta^4(x-z) G^{-1}(x,y)]$$

is a solution of Eq. (4.6).

Similarly, we assume Eq. (4.5) for $\partial_\nu' \Gamma_{\mu\nu}$, which is equivalent to Eq. (4.4), and substitute it into the equation for $\partial_\nu' \Gamma_{\mu\nu}$:

$$\partial_\nu' \Lambda_{\mu\nu} = -(\tau_3 \partial_\nu' L_{\mu\nu} \tau_3) V. \tag{4.7}$$

The verification of consistency is equally straightforward.

In this way we have shown that the self-energy equation, in particular its expanded form (2.12), has Ward identities as solutions. So we can be sure that there are gauge-invariant solutions for Γ_μ and $\Gamma_{\mu\nu}$. To the extent that the integral equations for Γ_μ and $\Gamma_{\mu\nu}$ are assumed to have unique solutions, gauge invariance is then guaranteed.

The fact that the superconductive solution does actually lead to gauge-invariant results was shown in the previous paper I explicitly for the vertex Γ_μ. It was found that there exist collective excitations of the sound-wave type which are strongly coupled to the longitudinal part of the vertex, and thereby forces the Ward identity (or current conservation) to be satisfied. Essentially the same argument will go through for $\Gamma_{\mu\nu}$.

According to the earlier work, I, $\Gamma_\mu(p',p)$ has the form (assuming no Coulomb effect which eliminates the low-lying collective modes)

$$\Gamma_i(p',p) = [(p+p')_i/2m] + [2i\tau_2 \alpha^2 \phi q_i / (q_0^2 - \alpha^2 \mathbf{q}^2)],$$
$$\Gamma_0(p',p) = \tau_3 + [2i\tau_2 \phi q_0 / (q_0^2 - \alpha^2 \mathbf{q}^2)],$$

where the second term is coupled to the collective states with the dispersion law $q_0^2 = \alpha^2 \mathbf{q}^2$. Substituting this into Eq. (4.5) we find that there must be in $\partial_\nu' \Gamma_{\mu\nu}$ a term proportional to τ_1 and coupled to the collective states. Since the collective states must contribute to each of the vertices in $\Gamma_{\mu\nu}$, we may assume that $\Gamma_{\mu\nu}(p,p;q,-q)$ contains a term $-4\tau_1 \phi h_\mu h_\nu / (q_0^2 - \alpha^2 q^2)^2$ where $h_\mu = (\alpha^2 q_i, q_4)$, so that this will match the contribution from the collective term in Γ_μ when inserted in the Ward identity (4.5). Actually, the τ_2 term in Γ_μ should contain also contributions from the noncollective part which does not become singular as $q_0, q \to 0$. This implies that $\Gamma_{\mu\nu}$ will, in general, have terms $\propto \tau_1 h_\mu h_\nu / (q_0^2 - \alpha^2 q^2)$ and $\tau_1 h_\mu h_\nu$. In fact, we have found that Γ_{ij} has a term $\propto \tau_1 \delta_{ij} q^2$ which would produce the field dependence of the energy gap. Such a term will be combined with the longitudinal term into $\tau_1(\delta_{ij} q^2 - q_i q_j)$ to satisfy gauge invariance, although we have not verified this by explicit calculation.

5. RELATION TO THE SIMPLE VARIATIONAL METHOD

The results obtained in Sec. 3 about the magnetic field dependence of the energy gap can also be derived, in fact in a simpler way, from the variational method originally used in the BCS paper. What we have to do is to minimize the total energy of the system in a magnetic field with respect to the energy gap parameter.[11] The energy change due to the magnetic field is

$$E_A = -\frac{1}{2} \sum_q \frac{K(q)}{1 - K(q)/q^2} |A_i^{\mathrm{ex}}(q)|^2, \tag{5.1}$$

to the second order in $\mathbf{A}^{\mathrm{ex}}$. Here $K(q)$ is the kernel in the London-Pippard relation

$$j_i(q) = +K(q) A_i(q). \tag{5.2}$$

The total field $\mathbf{A}$ is related to the external field $\mathbf{A}^{\mathrm{ex}}$ by

$$A_i(q) = A_i^{\mathrm{ex}}(q) / [1 - K(q)/q^2]. \tag{5.3}$$

The kernel K has the form

$$K(q) = -(1/\Lambda) f(q^2), \quad 1/\Lambda = ne^2/m. \tag{5.4}$$

The form factor $f(q)$ describes the nonlocal nature of the relation between j and A with a characteristic coherence length $\xi_0 = v_F / \pi \phi$. In other words,

$$f(q) = 1 - C(\xi_0 q)^2 + O((q\xi_0)^4). \tag{5.5}$$

The dependence of E_A on the gap parameter ϕ enters then through the coherence length ξ_0 only. There will be no field dependence of the energy gap in the London limit.

The total energy $E = E_0 + E_A$ of the system is then minimized with respect to ϕ. Let ϕ_0 be the solution which minimizes E_0 so that $\phi = \phi_0 + \delta\phi$. Then

$$E = E_0(\phi_0) + E_A(\phi_0) + \frac{1}{2} \frac{d^2 E_0(\phi_0)}{d\phi_0^2} (\delta\phi)^2 + \frac{dE_A(\phi_0)}{d\phi_0} \delta\phi, \tag{5.6}$$

since E_0 has a minimum at ϕ_0. The new minimum now is displaced by

$$\delta\phi = -\frac{dE_A}{d\phi_0} \Big/ \frac{d^2 E_0}{d\phi_0^2}. \tag{5.7}$$

With the Hartree wave function of BCS, $d^2E_0/d\phi_0^2$ is

[11] This has been carrried out by J. Bardeen (private communication).

easily calculated to be

$$\frac{d^2E_0}{d\phi_0^2}=\tfrac{1}{2}\sum_p\frac{\epsilon_p^2}{E_p^3}\sum_p\frac{\phi^2}{E_p^3}\bar{V}$$
$$=2N(1-\rho)\approx 2N;$$
$$\frac{dE_A}{d\phi_0}=\tfrac{1}{2}\sum_q\left[\frac{1}{1-K/q^2}\right]^2\frac{2C\Lambda^{-1}(q\xi_0)^2}{\phi_0}|\mathbf{A}^{\mathrm{ex}}(\mathbf{q})|^2 \qquad (5.8)$$
$$=\tfrac{1}{2}\sum_q 2C\Lambda^{-1}(q\xi_0)^2|\mathbf{A}(\mathbf{q})|^2;$$
$$N=mp_F/2\pi^2,\quad \Lambda^{-1}=e^2n/m=e^2p_F^3/3\pi^2m,\quad C=\pi^2/30.$$

Here the value of C is taken from Eq. (3.19). This yields

$$\delta(\phi^2)=-\sum_q[C(q\xi_0)^2/\Lambda N]|\mathbf{A}(\mathbf{q})|^2$$
$$=-\sum_q(\pi^2/45)e^2v_F^2(q\xi_0)^2|\mathbf{A}(\mathbf{q})|^2 \qquad (5.9)$$

which agrees with Eq. (3.12).

That this agreement is not accidental can be seen from the following observation. In Sec. 3 we have seen that the quantity $\delta\phi$ is the τ_1-proportional term in $\Sigma^{(2)}(p)/\rho$, where

$$\Sigma^{(2)}(p)=-\frac{1}{(2\pi)^4}\int V(p-l)\tau_3S(l)\tau_3d^4l,$$
$$S(l)=\tfrac{1}{2}\sum_q[G(l)\Gamma_\mu(-q)G(l+q)\Gamma_\nu(q)G(l)$$
$$+G(l)\Gamma_\nu(q)G(l-q)\Gamma_\mu(-q)G(l)]A_\mu(-q)A_\nu(q). \qquad (5.10)$$

Now take

$$\frac{1}{2}\frac{1}{(2\pi)^3}\int \mathrm{Tr}\frac{1}{2E_p}\tau_1\Sigma^{(2)}(\mathbf{p})d^3p$$
$$=\frac{1}{(2\pi)^3}(\rho\delta\phi)\int\frac{d^3p}{2E_p}=N\delta\phi, \qquad (5.11)$$

where the constancy of $\delta\phi$ near the Fermi surface is taken into account. On the other hand, the same quantity is equal to

$$-\frac{1}{2}\frac{1}{(2\pi)^7}\int\int \mathrm{Tr}\left[\tau_1\frac{1}{2E_p}V(p-l)\tau_3S(l)\tau_3\right]d^3pd^4l$$
$$=\frac{1}{2}\frac{1}{(2\pi)^4}\int \mathrm{Tr}[\tau_1S(l)]d^4l. \qquad (5.12)$$

In view of the definition of $S(l)$ and

$$G(l)\tau_1G(l)=i(d/d\phi)G(l),$$

this in turn is

$$=\frac{i}{4}\frac{1}{(2\pi)^4}\sum_q\left\{\frac{d}{d\phi}\int \mathrm{Tr}[\Gamma_\mu G(l+q)\Gamma_\nu G(l)]d^4l\right\}$$
$$\times A_\mu(-q)A_\nu(q),$$
$$=+\frac{1}{4}\sum_q\left\{\frac{d}{d\phi}K_{\mu\nu}(q)\right\}A_\mu(-q)A_\nu(q)\rightarrow$$
$$\frac{1}{4}\sum_q\left\{\frac{d}{d\phi}K(q)\right\}|A(q)|^2=-\frac{1}{2}\frac{d}{d\phi}E_A\bigg|_{A^{\mathrm{ex}}}, \qquad (5.13)$$

according to the definition of the kernel K. Note that the last line follows from Eqs. (5.1) and (5.3). Thus combining Eqs. (5.8), (5.11), and (5.13) we get

$$\delta\phi=-\frac{1}{2N}\frac{dE_A}{d\phi_0}=-\frac{dE_A}{d\phi_0}\bigg/\frac{d^2E_0}{d\phi_0^2},$$

in agreement with Eq. (5.7).

CONCLUDING REMARKS

Our results on the magnetic field dependence $\delta\phi$ of the energy gap, to the second order in the field, are given by Eqs. (3.12) and (3.25). They are in disagreement with the results obtained by Gupta and Mathur. A characteristic point of our formula is that $\delta\phi$ depends only on the field strength H for small q. According to Gupta and Mathur, on the other hand, $\delta\phi$ depends on the potential A. For large q, both results agree qualitatively, apart from different numerical coefficients.

Since these calculations have been done for an infinite medium, there remains the question about how to apply them to practical cases where thin films or colloidal particles have been used. It would seem appropriate to treat this more interesting problem in a separate paper.

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