Semiclassical Dispersion Theory of Interband Magneto-Optical Effects

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(Received July 23, 1962)

From the classical equation of motion a conductivity tensor is derived for a bound electron in a dc external magnetic field. Then the conductivity for a circularly polarized wave is obtained, which is expanded in terms of the magnetic field. With the appropriate form of the oscillator strength for the interband transitions, the conductivity components are evaluated for the zeroth, first, and second power of the magnetic field over the two energy bands for the direct and the indirect transitions. The results are used to obtain expressions for the interband Faraday rotation and the Voigt phase shift in the limits of $\omega<\omega_g$ and $\omega>\omega_g$, where ω is optical frequency and ω_g the frequency corresponding to the energy gap. In the latter case oscillatory behavior is described by the expression near the frequency of singularities with a loss term in the form of relaxation time τ.

INTRODUCTION

THE early interband magneto-optical phenomena investigated in semiconductors were the oscillatory magneto absorption where it was only necessary to calculate the absorption coefficient in the presence of dc magnetic field. This has been carried out by a number of investigators[1] by evaluating the absorption coefficient from the following integral:

$$\alpha(\omega,H)=\frac{K}{\omega}\sum_{nn'}\int dk_z dk_z' \times|M|^2\delta(E_{n'}(k')-E_n(k)-\hbar\omega), \quad (1)$$

where K is a constant, M the momentum matrix, z the direction of the magnetic field, and $E_{n'}(k')$, $E_n(k)$ the eigenvalues in the presence of the magnetic field for the two sets of bands considered. This type of integral has been used to evaluate the coefficients for the direct allowed, the direct forbidden, and the indirect transitions and applied to interpret the experimental results in several semiconductors.[2] More recently, however, the investigations have been extended to reflection phenomena in semiconductors[3] and semimetals[4] as well as to the study of the interband Faraday rotation[5] in some of these materials. Theory of reflection from a metal in magnetic fields has been also treated by Dresselhaus and Dresselhaus.[6] Furthermore, it has occurred to us that the interband Voigt effect which has recently been observed experimentally[7] is amenable to analysis at the same time. In order to interpret these experiments, it was necessary to consider the dispersion associated with the interband transitions as well as the absorption. For the interband Faraday rotation this was first carried out by the use of the Kramers-Kronig relation which expresses the dispersive components in terms of the absorption.[8] Similar results were obtained by Suffczynski.[9] Subsequently, in this treatment and an earlier one involving the Kramers-Heisenberg integrals[8] with a magnetic field, only the scalar or diagonal quantities were evaluated corresponding to a rotating frame of reference. This, however, did not give the proper symmetry relations for the Faraday rotation. However, in order to treat properly the magnetoreflection and the interband Faraday rotation and the Voigt effects, it is necessary to formulate the Kramers-Heisenberg integrals in tensor form in the presence of a magnetic field. The results then do possess the correct symmetry properties.

* On leave of absence from the Institute of Physics, Polish Academy of Science, Warsaw, Poland.

† Operated with support from the Air Force through the Air Force Office of Scientific Research.

‡ Operated with support from the U. S. Army, Navy, and Air Force.

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CONDUCTIVITY TENSOR RELATIONS

If we consider an electron in the valence band as a classical oscillator with the oscillator frequency corresponding to that of the energy associated with the particular interband transitions involved, then in the presence of a magnetic field we can write the equations of motion of this bound electron as follows:

$$\frac{d^2\mathbf{r}_k}{dt^2}+\omega_k^2\mathbf{r}_k+\nu_k\frac{d\mathbf{r}_k}{dt}+\frac{d\mathbf{r}_k}{dt}\times\boldsymbol{\omega}_c=\frac{e\mathbf{E}}{m}e^{i\omega t}. \qquad (2)$$

ω_k is the appropriate oscillator frequency; $\mathbf{r}_k$ is the displacement vector; $\mathbf{E}$ is the electric vector of the optical radiation; $\boldsymbol{\omega}_c=e\mathbf{H}/mc$ is the cyclotron frequency; $\mathbf{H}$ is the dc magnetic field; and $\nu_k d\mathbf{r}_k/dt$ is the damping term, where ν_k is the collision frequency. The index k corresponds to the wave number of the electron. This equation can be solved for the velocity components which are expressed in terms of the electric field components and those of the magnetic field. If the coordinate system is so chosen that the magnetic field is taken along the z direction, then we can calculate the current vector for all these transitions as follows:

$$\mathbf{J}=\sum_k e\mathbf{V}_k N_k, \qquad (3)$$

where $\mathbf{V}_k$ is the velocity of the electron and N_k is the number of transitions corresponding to the wave number k, which depends on the oscillator strength. Substituting the solution of Eq. (2) into Eq. (3), we can derive

$$\mathbf{J}=\hat{\sigma}\mathbf{E}, \qquad (4)$$

where $\hat{\sigma}$ is the complex conductivity tensor, components of which are given by

$$\sigma_{xx}=\sigma_{yy}=\tfrac{1}{2}(\sigma_+ + \sigma_-),$$
$$\sigma_{xy}=-\sigma_{yx}=\tfrac{1}{2}[\sigma_+ - \sigma_-], \quad \sigma_{zz}=\sigma_0. \qquad (5)$$

σ_0 is the conductivity in the absence of the magnetic field. The definition of the conductivities for circularly polarized waves are quantities which follow from Maxwell's equations when the rotating vectors are introduced. This is done by putting the equations of motion in component form and then combining the x and y components to obtain the solution for $J_x\pm iJ_y$ to obtain $\sigma_\pm$. It also follows from Maxwell's equations that $\sigma_\pm=\sigma_{xx}\mp i\sigma_{xy}$.

When this is done, the results become

$$\sigma_\pm=\frac{e^2}{m}\sum_k\frac{i\omega N_k}{\omega_k^2-\omega^2\pm\omega\omega_c+i\omega\nu_k}. \qquad (6)$$

Then

$$\sigma_{xx}=\frac{e^2}{m}\sum_k\frac{i\omega N_k(\omega_k^2-\omega^2-i\omega\nu_k)}{(\omega_k^2-\omega^2+i\omega\nu_k)^2-\omega^2\omega_c^2}, \qquad (7)$$

$$\sigma_{xy}=\frac{e^2}{m}\sum_k\frac{\omega^2\omega_c N_k}{(\omega_k^2-\omega^2+i\omega\nu_k)^2-\omega^2\omega_c^2}, \qquad (8)$$

$$\sigma_{zz}=\frac{e^2}{m}\sum_k\frac{i\omega N_k}{\omega_k^2-\omega^2+i\omega\nu_k}. \qquad (9)$$

Mathematically, it is simpler to evaluate the interband transitions for $\sigma_\pm$ and then obtain σ_{xx} and σ_{xy} from these by the use of Eq. (5). To do this we shall further assume that $N_k^+=N_k^-=N_k^z$ which is consistent with the results of the quantum treatment to a first-order approximation for an isotropic band. For the purpose of this paper we shall further simplify our results in order to arrive at the appropriate expressions for the scalar or diagonal component of the magnetoabsorption which has been considered to date and to also calculate the Faraday rotation and the Voigt effect or phase shift between linearly polarized waves parallel and perpendicular to the magnetic field. Furthermore, initially we shall ignore the relaxation term ν_k and solve for the "lossless" case. Later the loss will be reintroduced at the appropriate place.

In the approximation that is appropriate for our problem, we shall assume that $\omega_c\ll\omega$ even for high fields and we can, therefore, expand the conductivity as a function of magnetic field or ω_c. If we do this for $\nu_k=0$, we obtain

$$\sigma_\pm=\frac{e^2}{m}\sum_k\left(\frac{i\omega N_k}{\omega_k^2-\omega^2}\mp\frac{i\omega^2\omega_c N_k}{(\omega_k^2-\omega^2)^2}+\frac{i\omega^3\omega_c^2 N_k}{(\omega_k^2-\omega^2)^3}\right). \qquad (10)$$

Then

$$\sigma_{xx}^{(0)}=\sigma_{yy}^{(0)}=\sigma_{zz}=\frac{e^2}{m}\sum_k\frac{i\omega N_k}{\omega_k^2-\omega^2}, \qquad (11)$$

$$\sigma_{xy}^{(1)}=-\sigma_{yx}^{(1)}=\frac{e^2}{m}\sum_k\frac{\omega^2\omega_c N_k}{(\omega_k^2-\omega^2)^2}, \qquad (12)$$

$$\sigma_{xx}^{(2)}=\sigma_{yy}^{(2)}=\frac{e^2}{m}\sum_k\frac{i\omega^3\omega_c^2 N_k}{(\omega_k^2-\omega^2)^3}. \qquad (13)$$

We shall show presently that the above expressions of Eqs. (11), (12), and (13) are the terms associated with the diagonal or scalar magnetoabsorption or magnetoreflection, the Faraday rotation, and the Voigt effect, respectively.

From Maxwell's equations it has been shown that the complex indices of refraction $(n-ik)$ can be represented in the most general form in mks units by an equation as follows:

$$(n-ik)^2=\kappa(1+\sigma_{\mathrm{eff}}/i\omega\epsilon), \qquad (14)$$

where κ is the dielectric constant, σ_{eff} is the appropriate conductivity for longitudinal or transverse propagation, i.e., the Poynting vector is parallel or perpendicular to the magnetic field and ϵ is the permittivity. If we assume further that the conduction current is small compared to the displacement current, i.e., $\sigma_{\mathrm{eff}}<\omega\epsilon$, then we can write

$$n-ik\simeq\kappa^{1/2}(1+\sigma_{\mathrm{eff}}/2i\omega\epsilon). \qquad (15)$$

For longitudinal propagation,

$$\sigma_{\mathrm{eff}}=\sigma_\pm=\sigma_{xx}\mp i\sigma_{xy}. \qquad (16)$$

For transverse propagation,

$$\sigma_{\text{eff}}=\sigma_{\perp}=\sigma_{xx}+\sigma_{xy}^2/(i\omega\epsilon+\sigma_{xx})\simeq\sigma_{xx} \quad \text{for} \quad \mathbf{E}\perp\mathbf{H}, \tag{17}$$

since σ_{xx}, $\sigma_{xy}\ll\omega\epsilon$. Also,

$$\sigma_{\text{eff}}=\sigma_{\|}=\sigma_{zz} \quad \text{for} \quad \mathbf{E}\|\mathbf{H}. \tag{18}$$

From the definition of the Faraday effect, the angle of rotation per unit length θ is given by

$$\theta=(\omega/2c)(n_+-n_-). \tag{19}$$

Substituting the results of Eqs. (12), (15), and (16) into Eq. (19), we obtain

$$\theta=-\frac{\kappa^{1/2}}{2c\epsilon}\sigma_{xy}^{(1)}=-\frac{\kappa^{1/2}e^2}{2c\epsilon m}\sum_k\frac{\omega^2\omega_cN_k}{(\omega_k^2-\omega^2)^2}. \tag{20}$$

Similarly the phase shift of the Voigt effect is given by

$$\delta=(\omega/c)(n_{\|}-n_{\perp}). \tag{21}$$

From Eqs. (15), (17), (18), (11), and (13), we obtain the following:

$$\delta=-\frac{\kappa^{1/2}}{2ic\epsilon}\sigma_{xx}^{(2)}=-\frac{\kappa^{1/2}e^2}{2c\epsilon m}\sum_k\frac{\omega^3\omega_c^2N_k}{(\omega_k^2-\omega^2)^3}. \tag{22}$$

DIRECT TRANSITION

In evaluating the various quantities for the experimental situations under consideration, we shall separate the phenomena according to the situation which involves photon energies below the energy gap or those above, which usually involve oscillatory phenomena. When we do this, we have to consider the proper representation of the oscillator strength for the interband transition. It has been shown that this takes the form for the direct transition,

$$f_{kk'}=(1/m)(|M_{kk'}|^2/\hbar\omega_{kk'}), \tag{23}$$

where $M_{kk'}$ is the momentum matrix and is defined by

$$M_{kk'}=\langle k|\mathbf{p}\cdot\mathbf{e}|k'\rangle, \tag{24}$$

where $\mathbf{e}$ is the unit vector along the electric field of the electromagnetic wave. $\omega_{kk'}$ in the absence of a magnetic field is given by

$$\omega_{kk'}=\omega_g+\hbar k^2/2\mu=\omega_k, \tag{25}$$

where ω_g is the frequency corresponding to the energy gap, μ the reduced effective mass for two simple parabolic bands. When the quantization of the energy bands is considered in the presence of a magnetic field, then the energies corresponding to the transverse electric dipole transitions, i.e., for the circularly polarized fields, are represented by

$$\omega_n^{\pm}=\omega_g+(n+\tfrac{1}{2})\omega_c^{\star}\pm\gamma H+\hbar k_z^2/2\mu. \tag{26}$$

where $\omega_c^{\star}$ is the cyclotron frequency $\omega_c^{\star}=eH/\mu c$, γH is the splitting of the nth state due to the magnetic field, and γ is a phenomenological factor which involves parameters of the conduction and valence bands and may be represented in terms of equivalent or effective g factors which have an orbital origin and can have anomalously large values and reversal in sign in the presence of spin-orbit coupling. It can be given by

$$\gamma=\tfrac{1}{2}\mu_B(g_v+g_c), \tag{27}$$

where μ_B is the Bohr magneton, g_c and g_v are the effective g factors of the conduction and valence bands suitable to the splittings of the magnetic levels. Since this splitting is orbital in nature, the orbital splitting of the classical bound oscillators which were given by ω_c has to be replaced in the effective mass or band picture by γH in Eqs. (12) and (13) for the conductivities and in the evaluation of the expressions for the Faraday rotation. In order to sum up the contributions of all the electrons, it is assumed that the valence band is completely filled and the conduction band is empty. $N_k=f_{kk'}\Delta V$ the number of transitions becomes a product of the oscillator strength and the combined density of states, $\Delta V=2\pi^{-3}d^3k$ for the two bands. Then the summation with respect to k and k' can be replaced by an integral over the bands since the states are quasi-continuous. When all these steps are incorporated into the theory, the expressions for the three conductivities of interest for the two cases considered become

$$\sigma_{xx}^{(0)}=A\int\frac{|M_{kk'}|^2dV}{\omega_{kk'}(\omega_{kk'}^2-\omega^2)}=AI_0, \tag{28a}$$

$$\sigma_{xy}^{(1)}=\frac{\omega\omega_c}{i}A\int\frac{|M_{kk'}|^2dV}{\omega_{kk'}(\omega_{kk'}^2-\omega^2)}=A\frac{\omega\omega_c}{i}I_1, \tag{28b}$$

and

$$\sigma_{xx}^{(2)}=(\omega\omega_c)^2A\int\frac{|M_{kk'}|^2dV}{\omega_{kk'}(\omega_{kk'}^2-\omega^2)^3}=(\omega\omega_c)^2AI_2, \tag{28c}$$

where

$$A=i\omega e^2/m^2\hbar.$$

In order to evaluate the above integrals, the computations are facilitated by the following relations which are satisfied by I_0, I_1, and I_2:

$$I_1=\frac{1}{2\omega}\frac{dI_0}{d\omega}, \tag{29}$$

$$I_2=\frac{1}{4\omega}\frac{dI_1}{d\omega}=\frac{1}{8\omega^2}\left(\frac{d^2I_0}{d\omega^2}-\frac{1}{\omega}\frac{dI_0}{d\omega}\right). \tag{30}$$

Using Eqs. (29) and (30) and the expression given by Eqs. (20), (22), and (28), it follows that[10]

$$\theta=-\frac{\omega}{2c}\gamma H\frac{dn}{d\omega}, \tag{31}$$

and

$$\delta=-\frac{\omega}{8c}(\gamma H)^2\left(\frac{d^2n}{d\omega^2}-\frac{1}{\omega}\frac{dn}{d\omega}\right). \tag{32}$$

[10] The classical correspondence of Eq. (31) is given in T. S. Moss, *Optical Properties of Semiconductors* (Academic Press Inc., New York, 1959), Chap. 5, p. 85.

For photon energies below the gap, the density of states for spherical surfaces is

$$dv=[2/(2\pi)^3]d^3k=(1/\pi^2)k^2dk,\qquad \omega_{kk'}=\omega_g+\hbar k^2/2\mu. \tag{33}$$

For photon energies above the gap, the density of states is

$$dv=(\mu/\pi^2\hbar)\omega_c^\star dk_z,\qquad \omega_{kk'}=\omega_g+(n+\tfrac{1}{2})\omega_c^\star+\hbar k_z^2/2\mu. \tag{34}$$

A. $\omega<\omega_g$

If the integrals are carried out for the case of Eq. (33), the expressions for the appropriate quantities for the magnetoabsorption and magnetoreflection, Faraday rotation, and Voigt effect become

$$n=\kappa^{1/2}+(B/\omega^2)[2\omega_g^{1/2}-(\omega_g-\omega)^{1/2}-(\omega_g+\omega)^{1/2}], \tag{35a}$$

$$\theta=\frac{B\gamma H}{4c\omega}\left\{(\omega_g+\omega)^{-1/2}-(\omega_g-\omega)^{-1/2}+\frac{4}{\omega}[2\omega_g^{1/2}-(\omega_g-\omega)^{1/2}-(\omega_g+\omega)^{1/2}]\right\} \tag{35b}$$

$$\delta=-\frac{B(\gamma H)^2}{32c\omega}\left\{(\omega_g-\omega)^{-3/2}+(\omega_g+\omega)^{-3/2}+\frac{10}{\omega}[(\omega_g+\omega)^{-1/2}-(\omega_g-\omega)^{-1/2}]+\frac{32}{\omega^2}[2\omega_g^{1/2}-(\omega_g-\omega)^{1/2}-(\omega_g+\omega)^{1/2}]\right\}, \tag{35c}$$

where

$$B=\frac{e^2(2\mu)^{3/2}|M_{kk'}|^2\kappa^{1/2}}{8\pi m^2\epsilon\hbar^{3/2}}.$$

These results apply to those experiments in a magnetic field in which the frequency of photons is below the energy gap. In particular for low frequencies, we can show that

$$n=\kappa^{1/2}+B/4\omega_g^{3/2},\qquad \theta=-\frac{5B\gamma H\omega^2}{64c\omega_g^{7/2}},\quad \delta=\frac{B(\gamma H)^2\omega^3}{512c\omega_g^{11/2}}. \tag{36}$$

The first result of Eq. (36) shows that the index of refraction at low frequencies is essentially a sum of terms similar to the dispersive tail of the single idealized direct transition represented here. When the density of states is large as in the case of the higher transitions in germanium, silicon, etc., near 2 eV and 4 eV, then the coefficient B is large. These are the transitions which then determine the dielectric constant in the infrared and lower frequencies. The Faraday rotation on the other hand shows a dependence on ω^2, the square of the frequency, or λ^{-2} dependence. The Voigt effect, in turn, shows an ω^3 or λ^{-3} dependence. These results can be directly obtained from the integrals of Eqs. (28b) and (28c) by letting $\omega^2>0$ in the denominator. It is also apparent that the low-frequency Faraday rotation in most semiconductors is primarily due to the contribution of direct transitions with low energy gaps.

B. $\omega>\omega_g$

When the photon energies exceed that of the energy gap, then the transitions are quantized and show an oscillatory behavior. The expressions corresponding to those of Eqs. (35a), (35b), and (35c) become[11]

$$n=\kappa^{1/2}-\frac{B\omega_c^\star}{2\omega^2}\sum_n[2\omega_n^{-1/2}-(\omega_n-\omega)^{-1/2}-(\omega_n+\omega)^{-1/2}], \tag{37}$$

$$\theta=\frac{B\gamma H\omega_c^\star}{8c\omega}\sum_n\left\{(\omega_n+\omega)^{-3/2}-(\omega_n-\omega)^{-3/2}+\frac{4}{\omega}[-2\omega_n^{-1/2}+(\omega_n-\omega)^{-1/2}+(\omega_n+\omega)^{-1/2}]\right\} \tag{38a}$$

$$\delta=-\frac{3B(\gamma H)^2\omega_c^\star}{64c\omega}\sum_n\left\{(\omega_n-\omega)^{-5/2}+(\omega_n-\omega)^{-5/2}+\frac{10}{3\omega}[(\omega_n+\omega)^{-3/2}-(\omega_n-\omega)^{-3/2}]+\frac{32}{3\omega^2}[-2\omega_n^{-1/2}+(\omega_n-\omega)^{-1/2}+(\omega_n+\omega)^{-1/2}]\right\}, \tag{38b}$$

where

$$\omega_n=\omega_g+(n+\tfrac{1}{2})\omega_c^\star,\quad \omega_c^\star=eH/\mu c.$$

Each of the above expressions show singularities at the transitions between the magnetic levels of the conduction and valence bands. Actually in this region the most significant terms are those with singularities and in the narrow region above the energy gap the other terms can then be neglected. Then in order to properly represent the line shapes, it is necessary to introduce the damping term with a relaxation time. Hence,

$$n=\kappa^{1/2}+\frac{B\omega_c^\star}{2\omega^2}\sum_n\tau_n^{1/2}\,\mathrm{Re}(X_n+i)^{-1/2} \tag{39a}$$

$$\theta=\frac{B\gamma H\omega_c^\star}{4c\omega}\sum_n\tau_n^{3/2}\frac{d}{dX_n}\mathrm{Re}(X_n+i)^{-1/2}, \tag{39b}$$

$$\delta=\frac{-B(\gamma H)^2\omega_c^\star}{16c\omega}\sum_n\tau_n^{5/2}\frac{d^2}{dX_n^2}\mathrm{Re}(X_n+i)^{-1/2} \tag{39c}$$

[11] Expression (37) for the diagonal term has also been obtained by L. I. Korovin, Soviet Phys.—Solid State **3**, 1299 (1961).

where $X_n=(\omega_n-\omega)\tau_n$ and we have neglected the first derivative in Eq. (32), since that term is divided by ω and is, therefore, small compared to the term with the second derivative.

INDIRECT TRANSITION

The integrals for evaluating the indirect transition differ from that of the direct case in that they have to be evaluated over both the conduction and valence bands separately since the momenta are not conserved in the process of emitting or absorbing a phonon. In this case then, the double integrals take the form:

$$\sigma_{xx}^{(0)}=A\int\int\frac{|M_{kk'}|^2dvdv'}{\omega_{kk'}(\omega_{kk'}^2-\omega^2)},\tag{40a}$$

$$\sigma_{xy}^{(1)}=\frac{\omega\omega_c}{i}A\int\int\frac{|M_{kk'}|^2dvdv'}{\omega_{kk'}(\omega_{kk'}^2-\omega^2)^2},\tag{40b}$$

$$\sigma_{xx}^{(2)}=(\omega\omega_c)^2A\int\int\frac{|M_{kk'}|^2dvdv'}{\omega_{kk'}(\omega_{kk'}^2-\omega^2)^3},\tag{40c}$$

where $M_{kk'}$ is now a product of a momentum matrix for an allowed transition to an intermediate state and a phonon matrix from the intermediate to the final state. The phonon contribution is weighted by the appropriate phonon density factor. For energies below the gap, we have

$$dv=(1/\pi^2)k^2dk,\quad dv'=(1/\pi^2)k'^2dk',$$

and

$$\omega_{kk'}=\omega_g+\hbar k^2/2m_v+\hbar k'^2/2m_c\pm\omega_{\mathrm{ph}},$$

where ω_{ph} is the phonon frequency. For energies above the gap where the quantization of both the conduction and valence bands have to be taken into account, we have to sum the integrals of both sets of one-dimensional bands corresponding to the magnetic quantum numbers n and n'. In addition, the density of states factor then becomes

$$\begin{aligned}dv&=(m_v/\pi^2\hbar)\omega_{c1}^\star dk_z;\quad \omega_{c1}^\star=eH/m_vc,\\ dv'&=(m_c/\pi^2\hbar)\omega_{c2}^\star dk_z';\quad \omega_{c2}^\star=eH/m_cc,\end{aligned}\tag{41}$$

and

$$\begin{aligned}\omega_{kk'}=\omega_g\pm\omega_{\mathrm{ph}}+(n+\tfrac{1}{2})\omega_{c1}^\star+(n'+\tfrac{1}{2})\omega_{c2}^\star&\\ +\hbar k_z^2/2m_v+\hbar k_z'^2/2m_c.&\end{aligned}\tag{42}$$

When these are inserted in the expression for the integrals of Eq. (34), we obtain the following results:

For $\omega<\omega_g\pm\omega_{\mathrm{ph}}$,

$$\theta=\frac{D\gamma H}{2c}\left[\frac{\omega_g^2}{\omega^2}\ln\frac{\omega_g^2}{\omega^2-\omega^2}+\frac{\omega_g}{\omega}\ln\frac{\omega_g-\omega}{\omega_g+\omega}+1\right],\tag{43a}$$

$$\delta=\frac{D(\gamma H)^2}{4c\omega}\left[2\frac{\omega_g^2}{\omega^2}\ln\frac{\omega_g^2-\omega^2}{\omega_g^2}+\frac{3}{2}\frac{\omega_g}{\omega}\ln\frac{\omega_g+\omega}{\omega_g-\omega}+1\right].\tag{43b}$$

For very low frequency:

$$\theta=-D\gamma H\omega^2/12c\omega_g^2,\tag{43c}$$

$$\delta=-D(\gamma H)^2\omega^3/6c\omega_g^4,\tag{43d}$$

where

$$D=\frac{e^2(m_vm_c)^{3/2}|M_{kk'}|^2\kappa^{1/2}}{8\pi^3m^2\hbar^4\epsilon}.$$

For photon energies above the gap, $\omega>\omega_g\pm\omega_{\mathrm{ph}}$,

$$n=\kappa^{1/2}+\frac{D\omega_{c1}^\star\omega_{c2}^\star}{\omega^2}\sum_{nn'}\ln\frac{\omega_{nn'}^2}{\omega_{nn'}^2-\omega^2},\tag{44a}$$

$$\theta=-\frac{D\gamma H}{c}\frac{\omega_{c1}^\star\omega_{c2}^\star}{\omega^2}\sum_{nn'}\left(\ln\frac{\omega_{nn'}^2-\omega^2}{\omega_{nn'}^2}+\frac{\omega^2}{\omega_{nn'}^2-\omega^2}\right),\tag{44b}$$

$$\delta=\frac{D(\gamma H)^2}{4c}\frac{\omega_{c1}^\star\omega_{c2}^\star}{\omega^2}\sum_{nn'}\left(5\ln\frac{\omega_{nn'}^2-\omega^2}{\omega_{nn'}^2}+\frac{4\omega^2}{\omega_{nn'}^2-\omega^2}-\frac{2\omega^4}{(\omega_{nn'}^2-\omega^2)^2}\right),\tag{44c}$$

where

$$\omega_{nn'}=\omega_g+(n+\tfrac{1}{2})\omega_{c1}^\star+(n'+\tfrac{1}{2})\omega_{c2}^\star\pm\omega_{\mathrm{ph}}.$$

These results also indicate an oscillatory behavior since the logarithmic functions show singularities near the energy gap or the energies corresponding to the transitions between the Landau levels. Consequently, near the singularities it is appropriate to introduce a relaxation time $\tau_{nn'}$. If only those terms with singularities are retained, the Faraday rotation and the Voigt phase shift in the oscillatory case can be represented by the following expressions[5,7]:

$$n=\kappa^{1/2}-\frac{D\omega_{c1}^\star\omega_{c2}^\star}{\omega^2}\sum_{nn'}\mathrm{Re}\ln(X_{nn'}+i),\tag{45a}$$

$$\theta=-\frac{D\gamma H\omega_{c1}^\star\omega_{c2}^\star}{2c\omega}\sum_{nn'}\tau_{nn'}\frac{d}{dX_{nn'}}\mathrm{Re}\ln(X_{nn'}+i),\tag{45b}$$

$$\delta=\frac{D(\gamma H)^2\omega_{c1}^\star\omega_{c2}^\star}{8c\omega}\sum_{nn'}\tau_{nn'}^2\frac{d^2}{dX_{nn'}^2}\times\mathrm{Re}\ln(X_{nn'}+i),\tag{45c}$$

where

$$\mathrm{Re}\ln(X_{nn'}+i)=\tfrac{1}{2}\ln(X_{nn'}^2+1).$$

FORBIDDEN TRANSITION

The treatment for the direct forbidden transition becomes much more complicated than that for either the direct or the indirect case, particularly so for the quantized oscillatory situation. The quantum mechanical analysis has been carried out on the magnetoabsorption[1] and the diagonal components of the conductivity tensor have been derived.[8] At present the

experimental situation of interest corresponds only to the nonoscillatory Faraday rotation near the energy gap.[12] Hence, with the approximation $|M_{kk'}|^2 \approx |M_1|^2 k^2$ in Eq. (28), the Faraday rotation is given by

$$\theta = -\frac{C\gamma H}{\omega}\left\{ (\omega_g+\omega)^{1/2} - (\omega_g-\omega)^{1/2} + \frac{4}{3\omega}\left[2\omega_g^{3/2} - (\omega_g-\omega)^{3/2} - (\omega_g+\omega)^{3/2}\right]\right\}, \quad (46)$$

where

$$C = \frac{e^2|M_1|^2(2\mu)^{5/2}\kappa^{1/2}}{8\pi c\hbar^{7/2}m^2\epsilon}.$$

Expressions of this type with finite limits of integral can be evaluated to give curves similar to that given by Lax and Nishina[8] to account for the interband Faraday rotation of p-type Ge at low temperatures.

DISCUSSION

The semiclassical theory of the interband magneto-optical effects has been developed in this paper to give the proper mathematical form for the Faraday rotation and the other related phenomena. The main objective of this work has been to obtain the optical frequency dependence of these phenomena near the singularities where these effects are of the greatest experimental interest. Qualitatively near these singularities the present results do not differ greatly from the previous ones of Lax and Nishina. Mathematically, however, the latest results do obey the proper symmetry conditions. For mathematical simplicity, all of the expressions have been derived in terms of the idealized model of the two sets of simple bands with quadratic energy-momentum relations. Furthermore, in the limit of a weak magnetic field the dispersion relationship was expanded in power series of the magnetic field. This approximation is probably valid for fairly high values of the magnetic field. The semiclassical theory can be generalized without such expansion, but further complexities are certainly expected.

In the present work the orbital contribution was represented by the parameter γH in Eq. (26) which described the splitting of the quantized levels in a magnetic field. This parameter, of course, depends on the band parameter of both conduction and valence bands. In an actual case it can be evaluated for each magnetic level. Such evaluation has been made for the direct transition by Roth.[1] However, no theoretical calculations have been made for the individual transitions for low quantum numbers where the effective g factors show anomalous properties as they have been observed experimentally by the oscillatory Faraday rotation and the Voigt effect.[5,7] The rigorous quantum mechanical treatment of the dispersion theory in a magnetic field becomes rather complex even for an idealized model. The theory apparently includes the terms[13,14] in addition to those obtained from the semiclassical treatments given here. In well-known semiconductors the bands are often degenerate and complex in their mathematical form. Consequently, the theory becomes even more involved. Boswarva, Howard, and Lidiard[15] have given a quantum mechanical treatment of the interband Faraday rotation which does not agree with the results of this paper, nor with those obtained by Roth,[13] nor those of Bennett and Stern.[14] Furthermore, their results suffer from the fact that in the limit of zero frequency the Faraday rotation does not vanish. Hence, they insert a correction term which they justify on this basis to satisfy the requirement on the low-frequency limit. Nevertheless, in principle, it should be possible in the limit of high quantum numbers to justify the semiclassical dispersion relations by the correspondence principle. This seems to be the case, since the expression in the integral of Eq. (28b) has the form identical to that derived by Rosenfeld[16] for the Faraday rotation between the discrete levels in the paramagnetic materials. Hence, our result of Eq. (35b) is merely a summation of quasi-continuous set of levels distributed over the bands. Furthermore, the expression evaluated for a parabolic band for the Faraday rotation does obey the criteria set by BHL[15] without any artificiality, namely, $\theta \propto \omega^2$ as $\omega \to 0$ and the proper Kramers-Kronig relations are satisfied.

In addition, the results given here appear to explain the principal effects observed experimentally and furthermore seem to account for the line shape of the oscillatory interband Faraday rotation[5] and the Voigt effect in Ge.[7] Also, the results given here are useful in making quantitative interpretation of the effective g factor for the individual transitions.

Note added in proof. Since this paper has been submitted for publication, we have carried out a quantum mechanical analysis of the problem following the approach of Boswarva, Howard, and Lidiard[15] with an important modification. We have shown that their results were incorrect. The quantum treatment in the limit of approximations developed in this paper, i.e., low frequency and small magnetic field, gives the same results as those obtained here for the Faraday rotation and very similar but slightly modified results for the Voigt effect. The details of this treatment will be subject to future publication.

ACKNOWLEDGMENTS

We want to thank Dr. L. M. Roth, Dr. H. J. Zeiger, and H. S. Bennett for helpful discussions in connection with this problem.

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