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Nucleon-Nucleon Diffraction Scattering

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A calculation of diffraction effects in nucleon-nucleon collisions is carried out by using the unitarity of the S matrix. In the diffraction approximation, a set of coupled integral equations for the Wolfenstein amplitudes is derived. The inhomogeneous terms are calculated by taking into account the intermediate state with one pion and two nucleons. For the inelastic matrix elements the modified one-pion-exchange expressions have been used. The integral equations are then solved by a numerical iterative procedure; the results are compared with experiments around 1 GeV. The shape of the experimental cross section is reproduced, but evidence is obtained that potential scattering is still important at these energies. A remarkable prediction is the appearance of a backward peak in neutron-proton scattering.

I. INTRODUCTION

EXPERIMENTS on proton-proton interaction in the 1–3 GeV energy region[1] show the following features.

(i) Below 1.3 GeV the inelastic channels are dominated by single-pion production, which can be reproduced theoretically by using the corrected one-pion-exchange approximation.[2]

(ii) Above 1.3 GeV double-pion production becomes important,[3] while triple production is still rather unimportant up to $\simeq 3$ GeV. Furthermore, the double-pion production is mainly peripheral.[3]

(iii) The elastic scattering exhibits forward and backward peaks of typical diffractive origin.[1]

The corrected one-pion-exchange approximation allows one to obtain the matrix elements for single- and double-pion production; we shall now briefly show that one can calculate the diffraction elastic scattering simply from the S-matrix unitarity condition.

Let ρ be a complete set of observables (e.g., total angular momentum, parity, and isospin). In a representation in which the S matrix is diagonal (with elements f_ρ), one has from unitarity

$$\mathrm{Im} f_\rho - |f_\rho|^2 = 0, \tag{1}$$

below the threshold for inelastic processes. It then follows that f_ρ in this energy region can be written in the form

$$f_\rho = (e^{2i\delta_\rho} - 1)/2i, \tag{2}$$

with δ_ρ real.

Above the inelastic threshold, condition (1) is no longer satisfied. This means that if Eq. (2) is still to be valid, δ_ρ must be complex.

$$\delta_\rho = \alpha_\rho + i\beta_\rho, \tag{3}$$

with real α_ρ, β_ρ.

Condition (1) must then be replaced by

$$\mathrm{Im} f_\rho - |f_\rho|^2 = (1 - e^{-4\beta_\rho})/4. \tag{4}$$

Thus the function of β_ρ on the right-hand side of Eq. (4) is the term contributed, through unitarity, by the inelastic channels.

If α_ρ were negligible in comparison with β_ρ, we would then have a method for calculating the elastic-scattering cross section from the unitarity condition. Negligible α_ρ means, physically, that the "potential" scattering is no longer important, a situation which is likely at high energy and which seems to be justified by the experimental diffraction shape of the elastic angular distributions. The approximation in which the α_ρ are neglected is termed the "diffraction approximation." It must be noted that putting $\alpha_\rho = 0$ means taking f_ρ as being purely imaginary. Therefore, any amplitude which can be used for describing N-N scattering can be taken as being purely imaginary in this approximation, since it is a linear function of the diagonal elements f_ρ. Of course, the validity of such an approximation becomes better and better for increasing energy. For this reason we restrict ourselves to energies above 900 MeV. In this energy region we can also neglect the deuteron production contribution. On the other hand, the number of open inelastic channels increases with energy, so that the calculation becomes extremely involved. We can take as an upper limit that energy at which triple-pion production is no longer negligible. As pointed out in (ii) this is about 3 GeV.

The advantage of our approach is in the fact that we do not assume any model for the inelastic processes. The inelastic matrix element can be considered as taken directly from experiments. The disadvantage is,

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[1] See, for instance: A. P. Batson, B. B. Culwick, J. G. Hill, and L. Riddiford, Proc. Roy. Soc. (London) **A251**, 218 (1959); W. J. Fickinger, E. Pickup, D. K. Robinson, and E. O. Salant, Phys. Rev. **125**, 2082 (1962); G. A. Smith, H. Courant, E. C. Fowler, H. Kraybill, J. Sandweiss, and H. Taft, *ibid.* **123**, 2160 (1961).
[2] E. Ferrari and F. Selleri, Phys. Rev. Letters **7**, 387 (1961).
[3] E. Pickup, D. K. Robinson, and E. O. Salant, Phys. Rev. **125**, 2091 (1962).

however, in the fact that we cannot calculate the real parts of the phase shifts.

A different approach has been suggested by Amati, Fubini, Stanghellini, and Tonin,[4] who proposed a general model for high-energy diffraction processes.

In Sec. II we derive from the unitarity condition for the S matrix a set of integral properties of the scalar amplitudes (Wolfenstein parameters) for nucleon-nucleon scattering. In Sec. III we discuss the general structure of the inelastic one-pion–two-nucleon contribution; it is calculated directly in Sec. IV by using the corrected one-pion-exchange expression for the inelastic matrix elements. Section V is devoted to the numerical evaluation of the Wolfenstein amplitudes and to comparison with experiment.

A forthcoming paper will take into account the two-pion–two-nucleon intermediate state; the calculation will be extended up to about 3 GeV.

II. THE UNITARITY CONDITION

We start from the following definition of the M-matrix element for the process $p_1+p_2\to q_1+\cdots+q_n$:

$$S_{\mathrm{fi}}=\delta_{\mathrm{fi}}+i(2\pi)^{4-\frac{3}{2}(n+2)}\Big[\prod_{i=1}^{f} m_i/2^b\Big]^{1/2}[p_{10}p_{20}\cdots q_{n0}]^{-1/2} \times\delta^4(q_1+\cdots+q_n-p_1-p_2)\mathfrak{M}_{\mathrm{fi}}. \quad (5)$$

n is the number of particles in the final state, $f(b)$ is the total number of fermions (bosons). Of course, $f+b=n+2$. Using the definition (5), the unitarity condition $S^\dagger S=1$ can be written for the case of elastic nucleon-nucleon scattering $p_1+p_2\to p_1'+p_2'$,

$$i\langle p_1s_1p_2s_2|M|p_1's_1'p_2's_2'\rangle^*-i\langle p_1's_1'p_2's_2'|M|p_1s_1p_2s_2\rangle$$
$$=\frac{m^2}{(2\pi)^2}\sum_{r_1r_2}\int_{\Omega/2}\frac{d^3q_1d^3q_2}{q_{10}q_{20}}\delta^4(p_1+p_2-q_1-q_2) \times\langle q_1r_1q_2r_2|M|p_1's_1'p_2's_2'\rangle^* \times\langle q_1r_1q_2r_2|M|p_1s_1p_2s_2\rangle+R, \quad (6)$$

where s_i, s_i', and r_i are spin variables, m is the nucleon mass, and R is the term contributed by the inelastic channels.

In this and a forthcoming paper, we perform a calculation of R taking into account single- and double-pion production, without deuteron formation. As explained in the introduction, this approximation is expected to be valid in the 1–3 GeV energy region. Our expression for R is thus

$$R=R_A+R_B, \quad (7)$$

where

$$R_A=\frac{m^2}{2(2\pi)^5}\sum_{r_1r_2\alpha}\int_{\Omega/2}\frac{d^3q_1d^3q_2d^3q}{q_{10}q_{20}q_0} \times\delta^4(p_1+p_2-q_1-q_2-q) \times\langle q_1r_1q_2r_2q\alpha|M|p_1's_1'p_2's_2'\rangle^* \times\langle q_1r_1q_2r_2q\alpha|M|p_1s_1p_2s_2\rangle, \quad (8)$$

$$R_B=\frac{m^2}{4(2\pi)^8}\sum_{r_1r_2\beta\gamma}\int_{\Omega/4}\frac{d^3q_1d^3q_2d^3qd^3q'}{q_{10}q_{20}q_0q_0'} \times\delta^4(p_1+p_2-q_1-q_2-q-q') \times\langle q_1r_1q_2r_2q\beta q'\gamma|M|p_1's_1'p_2's_2'\rangle^* \times\langle q_1r_1q_2r_2q\beta q'\gamma|M|p_1s_1p_2s_2\rangle. \quad (9)$$

In formulas (6) and (8) the integration domain is half of the possible phase space Ω, because of the identity of the nucleons q_1 and q_2. Because of the further identity of the two intermediate pions, the integration is over $\Omega/4$ in Eq. (9).

q and q' are the four-momenta of the pions in the intermediate state, and α, β, γ are their charge indices.

As is well known,[5] the M-matrix element for nucleon-nucleon scattering in the c.m. system can be written, under the hypotheses of parity conservation, time reversal invariance, and charge independence:

$$\langle p_1's_1'p_2's_2'|M|p_1s_1p_2s_2\rangle =\chi_{s_2'}^\dagger\chi_{s_1'}^\dagger[a+ic(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N})+m\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N} +d\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P}+f\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}]\chi_{s_2}\chi_{s_1}, \quad (10)$$

where the χ's are two-dimensional Pauli spinors, and $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$ are the usual spin matrices in the spinor space of nucleon 1 and nucleon 2, respectively. Furthermore

$$\mathbf{N}=\frac{\mathbf{p}_1\times\mathbf{p}_1'}{|\mathbf{p}_1\times\mathbf{p}_1'|};\quad \mathbf{P}=\frac{\mathbf{p}_1+\mathbf{p}_1'}{|\mathbf{p}_1+\mathbf{p}_1'|};\quad \mathbf{K}=\frac{\mathbf{p}_1-\mathbf{p}_1'}{|\mathbf{p}_1-\mathbf{p}_1'|}. \quad (11)$$

The scalar amplitudes a, c, m, d, f depend only on the total c.m. energy W and the scattering angle ϑ; *a priori* they can be complex. It should be noted that our d and f amplitudes are related to the g and h Wolfenstein parameters[5] by the following relations:

$$d=g+h;\quad f=g-h. \quad (12)$$

It is intuitively clear (and we shall prove this in the following) that each term in the right-hand side of Eq. (6) can also be written in a form analogous to Eq. (10). In particular, we shall call $2a_0,\cdots,2f_0$ the scalar amplitudes of the R term (the factor 2 is introduced for convenience). From the structure of Eq. (6) it then follows that the left-hand side will again have the form of Eq. (10) with the scalar amplitudes substituted by

[4] D. Amati, S. Fubini, A. Stanghellini, and M. Tonin, Nuovo cimento **22**, 569 (1961).

[5] See, for instance: H. P. Stapp, Ann. Rev. Nuclear Sci. **10**, 292 (1960).

twice their imaginary parts. Since the first term in the right-hand side of Eq. (6) is a quadratic functional of the same amplitudes, it follows that Eq. (6) can be transformed into a set of integral conditions for $a, \cdots f$, when $a_0, \cdots f_0$ are known.

We obtain

$$
\begin{aligned}
\mathrm{Im}a&=a_0+\frac{m^2}{(2\pi)^2}\frac{P}{4W}\int_\Omega d\Omega\,\tfrac{1}{4}\,\mathrm{Tr}[M''^*M'],\\
\mathrm{Im}c&=c_0+\frac{m^2}{(2\pi)^2}\frac{P}{8W}\int_\Omega d\Omega\,\tfrac{1}{4}\,\mathrm{Tr}[-i(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N})M''^*M'],\\
\mathrm{Im}m&=m_0+\frac{m^2}{(2\pi)^2}\frac{P}{4W}\int_\Omega d\Omega\,\tfrac{1}{4}\,\mathrm{Tr}[\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}M''^*M'],\\
\mathrm{Im}d&=d_0+\frac{m^2}{(2\pi)^2}\frac{P}{4W}\int_\Omega d\Omega\,\tfrac{1}{4}\,\mathrm{Tr}[\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P}M''^*M'],\\
\mathrm{Im}f&=f_0+\frac{m^2}{(2\pi)^2}\frac{P}{4W}\int_\Omega d\Omega\,\tfrac{1}{4}\,\mathrm{Tr}[\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}M''^*M'].
\end{aligned}
\tag{13}
$$

In Eq. (13) the integration is over the full solid angle and a factor $\frac{1}{2}$ has been put in front of the integral. Furthermore one has

$$
\begin{aligned}
M''^*=a''^*-ic''^*(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}''+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}'')
&+m''^*\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}''\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}''+d''^*\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}''\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P}''\\
&+f''^*\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}''\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}'',
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
M'=a'+ic'(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}'+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}')
&+m'\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}'\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}'+d'\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}'\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P}'\\
&+f'\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}'\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K}'.
\end{aligned}
\tag{15}
$$

In the above formulas we have put

$$
\mathbf{N}''=\frac{\mathbf{p}_1'\times\mathbf{q}_1}{|\mathbf{p}_1'\times\mathbf{q}_1|};\quad
\mathbf{P}''=\frac{\mathbf{p}_1'+\mathbf{q}_1}{|\mathbf{p}_1'+\mathbf{q}_1|};\quad
\mathbf{K}''=\frac{\mathbf{p}_1'-\mathbf{q}_1}{|\mathbf{p}_1'-\mathbf{q}_1|};
\tag{16}
$$

$$
\mathbf{N}'=\frac{\mathbf{p}_1\times\mathbf{q}_1}{|\mathbf{p}_1\times\mathbf{q}_1|};\quad
\mathbf{P}'=\frac{\mathbf{p}_1+\mathbf{q}_1}{|\mathbf{p}_1+\mathbf{q}_1|};\quad
\mathbf{K}'=\frac{\mathbf{p}_1-\mathbf{q}_1}{|\mathbf{p}_1-\mathbf{q}_1|}.
\tag{17}
$$

The primes on the amplitudes mean that these are functions of W and $\cos\Theta''$, and of W and $\cos\Theta'$, where Θ'' and Θ' are the angles $(\mathbf{p}_1',\mathbf{q}_1)$ and $(\mathbf{p}_1,\mathbf{q}_1)$, respectively.

It should be noted that the symbol Tr in Eq. (13) means a double trace, over both $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\sigma}^{(2)}$.

The evaluation of the traces in the right-hand side of Eq. (13) gives

$$
\begin{aligned}
\tfrac{1}{4}\,\mathrm{Tr}[M''^*M']=a''^*a'&+2c''^*c'\mathbf{N}''\cdot\mathbf{N}'+m''^*d'(\mathbf{N}''\cdot\mathbf{P}')^2+m''^*f'(\mathbf{N}''\cdot\mathbf{K}')^2+d''^*m'(\mathbf{P}''\cdot\mathbf{N}')^2+d''^*d'(\mathbf{P}''\cdot\mathbf{P}')^2\\
&+d''^*f'(\mathbf{P}''\cdot\mathbf{K}')^2+f''^*m'(\mathbf{K}''\cdot\mathbf{N}')^2+f''^*d'(\mathbf{K}''\cdot\mathbf{P}')^2+f''^*f'(\mathbf{K}''\cdot\mathbf{K}')^2+m''^*m'(\mathbf{N}''\cdot\mathbf{N}')^2,
\end{aligned}
\tag{18}
$$

$$
\begin{aligned}
&\tfrac{1}{4}\,\mathrm{Tr}[-i(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N})M''^*M']\\
&=2[a''^*c'\mathbf{N}\cdot\mathbf{N}'-c''^*a'\mathbf{N}\cdot\mathbf{N}''+c''^*c'\mathbf{N}\cdot\mathbf{N}''\times\mathbf{N}'-c''^*m'(\mathbf{N}\cdot\mathbf{N}')(\mathbf{N}''\cdot\mathbf{N}')-c''^*d'(\mathbf{N}\cdot\mathbf{P}')(\mathbf{N}''\cdot\mathbf{P}')\\
&\quad+m''^*c'(\mathbf{N}\cdot\mathbf{N}'')(\mathbf{N}'\cdot\mathbf{N}'')+d''^*c'(\mathbf{N}\cdot\mathbf{P}'')(\mathbf{N}'\cdot\mathbf{P}'')+f''^*c'(\mathbf{N}\cdot\mathbf{K}'')(\mathbf{N}'\cdot\mathbf{K}'')\\
&\quad-c''^*f'(\mathbf{N}\cdot\mathbf{K}')(\mathbf{N}''\cdot\mathbf{K}')+m''^*m'(\mathbf{N}\cdot\mathbf{N}''\times\mathbf{N}')(\mathbf{N}''\cdot\mathbf{N}')+m''^*d'(\mathbf{N}\cdot\mathbf{N}''\times\mathbf{P}')(\mathbf{N}''\cdot\mathbf{P}')\\
&\quad+m''^*f'(\mathbf{N}\cdot\mathbf{N}''\times\mathbf{K}')(\mathbf{N}''\cdot\mathbf{K}')+d''^*m'(\mathbf{N}\cdot\mathbf{P}''\times\mathbf{N}')(\mathbf{P}''\cdot\mathbf{N}')+d''^*d'(\mathbf{N}\cdot\mathbf{P}''\times\mathbf{P}')(\mathbf{P}''\cdot\mathbf{P}')\\
&\quad+d''^*f'(\mathbf{N}\cdot\mathbf{P}''\times\mathbf{K}')(\mathbf{P}''\cdot\mathbf{K}')+f''^*m'(\mathbf{N}\cdot\mathbf{K}''\times\mathbf{N}')(\mathbf{K}''\cdot\mathbf{N}')+f''^*d'(\mathbf{N}\cdot\mathbf{K}''\times\mathbf{P}')(\mathbf{K}''\cdot\mathbf{P}')\\
&\quad+f''^*f'(\mathbf{N}\cdot\mathbf{K}''\times\mathbf{K}')(\mathbf{K}''\cdot\mathbf{K}')],
\end{aligned}
\tag{19}
$$

$$
\begin{aligned}
&\tfrac{1}{4}\,\mathrm{Tr}[\boldsymbol{\sigma}^{(1)}\cdot\mathbf{Q}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{Q}M''^*M']\\
&=a''^*m'(\mathbf{Q}\cdot\mathbf{N}')^2+a''^*d'(\mathbf{Q}\cdot\mathbf{P}')^2+a''^*f'(\mathbf{Q}\cdot\mathbf{K}')^2+2c''^*c'(\mathbf{Q}\cdot\mathbf{N}'')(\mathbf{Q}\cdot\mathbf{N}')+2c''^*m'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{N}')(\mathbf{Q}\cdot\mathbf{N}')\\
&\quad+2c''^*d'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{P}')(\mathbf{Q}\cdot\mathbf{P}')+2c''^*f'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{K}')(\mathbf{Q}\cdot\mathbf{K}')+m''^*a'(\mathbf{Q}\cdot\mathbf{N}'')^2-2m''^*c'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{N}')(\mathbf{Q}\cdot\mathbf{N}'')\\
&\quad-m''^*m'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{N}')^2-m''^*d'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{P}')^2-m''^*f'(\mathbf{Q}\cdot\mathbf{N}''\times\mathbf{K}')^2+d''^*a'(\mathbf{Q}\cdot\mathbf{P}'')^2\\
&\quad-2d''^*c'(\mathbf{Q}\cdot\mathbf{P}''\times\mathbf{N}')(\mathbf{Q}\cdot\mathbf{P}'')-d''^*m'(\mathbf{Q}\cdot\mathbf{P}''\times\mathbf{N}')^2-d''^*d'(\mathbf{Q}\cdot\mathbf{P}''\times\mathbf{P}')^2-d''^*f'(\mathbf{Q}\cdot\mathbf{P}''\times\mathbf{K}')^2\\
&\quad+f''^*a'(\mathbf{Q}\cdot\mathbf{K}'')^2-2f''^*c'(\mathbf{Q}\cdot\mathbf{K}''\times\mathbf{N}')(\mathbf{Q}\cdot\mathbf{K}'')-f''^*m'(\mathbf{Q}\cdot\mathbf{K}''\times\mathbf{N}')^2\\
&\quad-f''^*d'(\mathbf{Q}\cdot\mathbf{K}''\times\mathbf{P}')^2-f''^*f'(\mathbf{Q}\cdot\mathbf{K}''\times\mathbf{K}')^2.
\end{aligned}
\tag{20}
$$

The last result can be used for the calculation of the right-hand side of the last three equations (13), by putting $\mathbf{Q}=\mathbf{N}, \mathbf{P}, \mathbf{K}$, respectively.

In general, the relations (13) do not constitute a uniquely soluble set of equations, since they contain ten unknown quantities (real and imaginary parts of the five amplitudes). As explained in the introduction, the diffraction approximation makes each amplitude purely imaginary. In this approximation (13) becomes a uniquely soluble set of equations and determines the amplitudes $a, \cdots, f$ in terms of $a_0, \cdots, f_0$.

The calculation of the latter quantities forms the object of the next two sections, while the practical solution of the system (13) is discussed in Sec. V.

III. INELASTIC CONTRIBUTION

In this section we discuss the inhomogeneous terms of Eq. (13), taking into account only the one-pion–two-nucleon intermediate state. The quantity to be evaluated is R_A as defined in Eq. (8).

In order to emphasize the effect of the identity of the nucleons, we split the matrix element into four contributions, as shown graphically in Fig. 1. The heavy lines represent any system which can be exchanged between the nucleons. Correspondingly, the matrix element is written

$$\mathfrak{M}=\langle q_1r_1q_2r_2q\alpha|M|p_1s_1p_2s_2\rangle=\mathfrak{M}_1+\mathfrak{M}_2-\mathfrak{M}_3-\mathfrak{M}_4. \quad (21)$$

We note the relations

$$\begin{aligned}\mathfrak{M}_2(p,q)&=\mathfrak{M}_1(\bar{p},\bar{q}),\\ \mathfrak{M}_3(p,q)&=\mathfrak{M}_1(\bar{p},q),\\ \mathfrak{M}_4(p,q)&=\mathfrak{M}_1(p,\bar{q}),\end{aligned} \quad (22)$$

which can easily be deduced by looking at Fig. 1. In the above formulas we have represented the momentum and spin variables in one symbol. Therefore, p stands for "$p_1, p_2; s_1, s_2$" and $\bar{p}$ means that the exchanges $p_1\leftrightarrow p_2$ *and* $s_1\leftrightarrow s_2$ have been performed. Similarly, q stands for q_1, q_2, q and r_1, r_2, α; $\bar{q}$ means the exchange of the nucleon variables.

Corresponding to the splitting (22), R_A splits into a sum of 16 contributions. More precisely, by defining

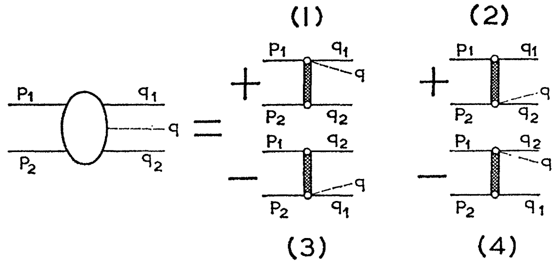

FIG. 1. The four contributions to the single-pion production M-matrix element. The heavy lines represent any system which can be exchanged between the nucleons.

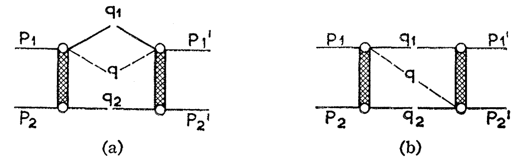

FIG. 2. Graphical representation of the single-pion production contributions to the unitarity condition.

$$\mathfrak{F}=\frac{m^2}{2(2\pi)^5}\sum_{r_1r_2\alpha}\int_{\Omega/2}\frac{d^3q_1d^3q_2d^3q}{q_{10}q_{20}q_0}\times\delta^4(p_1+p_2-q_1-q_2-q), \quad (23)$$

we obtain

$$\begin{aligned}R_A=R_{11}+R_{22}+R_{33}+R_{44}+R_{12}-R_{13}-R_{14}+R_{21}-R_{23}\\ -R_{24}-R_{31}-R_{32}+R_{34}-R_{41}-R_{42}+R_{43},\end{aligned} \quad (24)$$

where

$$R_{ij}(p',p)=\mathfrak{F}[\mathfrak{M}_i'^*(p',q)\mathfrak{M}_j(p,q)]. \quad (25)$$

By means of Eq. (22) we can reduce any R_{ij} to an expression containing $\mathfrak{M}_1'^*$ and $\mathfrak{M}_1$ only. Taking into account the fact that changing q into $\bar{q}$ in *both* the matrix elements in R_{ij} does not change R_{ij} itself, since this means exchanging the dummy variables $q_1\leftrightarrow q_2$ and $r_1\leftrightarrow r_2$, we get for R_A

$$\begin{aligned}R_A=&2R_{11}(p',p)+2R_{11}(\bar{p}',\bar{p})\\ &-2R_{11}(p',\bar{p})-2R_{11}(\bar{p}',p)\\ &-2R_{12}(p',p)-2R_{12}(\bar{p}',\bar{p})\\ &+2R_{12}(p',\bar{p})+2R_{12}(\bar{p}',p).\end{aligned} \quad (26)$$

The problem is thus reduced to calculation of only two terms, namely, R_{11} and R_{12}, and to the symmetrization of the result according to Eq. (26). The calculations which have to be performed are graphically indicated in Fig. 2.

The practical evaluation of R_{12} would be much more involved than that of R_{11}. There are fortunately good reasons for thinking that this term is small. Consider the diagram on the left of Fig. 2(b), and suppose for a moment that the pion q and the nucleon q_1 emerge in a state of fixed relative angular momentum, say J_0.

In the same diagram q and q_2 will come out in a state containing many angular momenta since no correlation between these particles is implied. However, this diagram must be multiplied with the one in the right, which implies correlation between q and q_2, and no correlation between q and q_1. Therefore, it is clear that only that part of the first (second) diagram having q and q_2 (q and q_1) in the state $J=J_0$ can contribute to R_{12}. Since only a small part of the two diagrams gives nonvanishing contributions, R_{12} must be small. This argument still applies when q and q_1 (q and q_2) are in a state containing many angular momenta in the first (second) diagram. A numerical check in the case of forward scattering, where R_{12} reduces to a mixed interference term for the total cross section of the

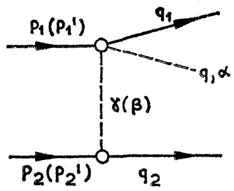

FIG. 3. Single-pion production according to the one-pion exchange model.

process of single-pion production, has been performed by Da Prato[6]. In the next section, therefore, we carry out the calculation of R_{11} alone.

IV. DERIVATION OF THE SCALAR AMPLITUDES

We now calculate the transition matrix elements appearing in Eq. (25), according to the modified one-pion-exchange model in which the exchanged system in Figs. 1 and 2 is a single π meson. The modifications consist in taking into account the vertex and propagator form factors as discussed in reference 2. The form factors that we use here are the same as adopted in reference 2, where it is shown that they reproduce the experimental data for single-pion production rather well. However, we wish to stress that the validity of our calculation does not depend on the correctness of the interpretation of these form factors as true pionic form factors of the nucleon. In other words, one could say that we use a phenomenological matrix element which reproduces the experimental data for the inelastic channel.

The graphs corresponding to the matrix elements $\mathfrak{M}_1$ ($\mathfrak{M}_1'$) are shown in Fig. 3, where γ (β) is the charge index of the virtual exchanged pion. The form of these matrix elements is

$$\mathfrak{M}_1=\bar{u}(q_2)G_r\gamma_5\tau^\gamma u(p_2)P(\Delta^2)\times\bar{u}(q_1)[-A^{\alpha\gamma}+i\gamma\cdot qB^{\alpha\gamma}]u(p_1),\quad(27)$$

$$\mathfrak{M}_1'=\bar{u}(q_2)G_r\gamma_5\tau^\beta u(p_2')P(\Delta'^2)\times\bar{u}(q_1)\,[-A'^{\alpha\beta}+i\gamma\cdot qB'^{\alpha\beta}]u(p_1'),\quad(28)$$

where

$$P(\Delta^2)=K(\Delta^2)K'(\Delta^2)/(\Delta^2+\mu^2),\quad(29)$$

and K, K' are the form factors of the vertex and propagator, respectively, and μ is the pion mass. The last factors in Eqs. (27) and (28) are the matrix elements for off-shell pion-nucleon scattering; they contain the invariant amplitudes A and B which depend on the following variables:

$$\begin{aligned}&A, B:\ \omega^2=-(q+q_1)^2,\quad t^2=(q_1-p_1)^2,\quad \Delta^2=(q_2-p_2)^2;\\&A', B':\ \omega^2=-(q+q_1)^2,\quad t'^2=(q_1-p_1')^2,\quad \Delta'^2=(q_2-p_2')^2.\end{aligned}\quad(30)$$

(We use the metric $a\cdot b=\mathbf{a}\cdot\mathbf{b}-a_0b_0$ and units $\hbar=c=1$.) Passing now to bi-dimensional spinors, *in the system* $\mathbf{q}+\mathbf{q}_1=0$, we get from Eqs. (27) and (28)

$$\begin{aligned}\mathfrak{M}_1&=\frac{\rho G_r}{2m}\chi_{r_2}{}^\dagger\tau^\gamma\left[\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}_2}{q_{20}+m}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}_2}{p_{20}+m}\right]\chi_{s_2}P(\Delta^2)\\&\quad\times\frac{4\pi\omega}{m}\chi_{r_1}{}^\dagger\left[f_1{}^{\alpha\gamma}+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_1\boldsymbol{\sigma}\cdot\mathbf{p}_1}{q_1p_1}f_2{}^{\alpha\gamma}\right]\chi_{s_1},\\\mathfrak{M}_1'^*&=\frac{\rho' G_r}{2m}\chi_{s_2'}{}^\dagger\tau^\beta\left[\frac{i\boldsymbol{\sigma}\cdot\mathbf{p}_2'}{p_{20}'+m}-\frac{i\boldsymbol{\sigma}\cdot\mathbf{q}_2}{q_{20}+m}\right]\chi_{r_2}P(\Delta'^2)\\&\quad\times\frac{4\pi\omega}{m}\chi_{s_1'}{}^\dagger\left[(f_1{}'^{\alpha\beta})^*+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{q}_1}{p_1'q_1}(f_2{}'^{\alpha\beta})^*\right]\chi_{r_1},\end{aligned}\quad(31)$$

where

$$\begin{aligned}\rho&=[(q_{20}+m)(p_{20}+m)]^{1/2};\\\rho'&=[(q_{20}+m)(p_{20}'+m)]^{1/2},\end{aligned}\quad(32)$$

and the amplitudes f_1 and f_2 have been defined by Ferrari and Selleri[7] for the off-shell case and depend on the variables (30). Multiplying together the two amplitudes (31) we first perform the summations over the discrete variables r_1, r_2, and α, as indicated in Eqs. (23) and (25). The summation over α is reported with some detail in Appendix 1, together with the summation over the charge indices β and γ of the intermediate pions. The result is

$$\begin{aligned}R_{11}&=\frac{f^2}{\pi^2\mu^2}\int_{\Omega/2}\frac{d^3q_2}{q_{20}}\rho\rho'\omega^2P(\Delta^2)P(\Delta'^2).\\&\chi_{s_2'}{}^\dagger\cdot\chi_{s_1'}{}^\dagger\left[\frac{q_{20}-m}{q_{20}+m}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2'\boldsymbol{\sigma}\cdot\mathbf{p}_2}{(p_{20}'+m)(p_{20}+m)}\right.\\&\left.-\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_2'\boldsymbol{\sigma}\cdot\mathbf{q}_2}{(p_{20}'+m)(q_{20}+m)}-\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{p}_2}{(q_{20}+m)(p_{20}+m)}\right]^{(2)}\\&\times\sum_{IT}P_I\alpha_{IT}J^{(T)}\chi_{s_2}\chi_{s_1},\end{aligned}\quad(33)$$

where the operator labeled with (2) has to act between spinors $\chi_{s_2'}{}^\dagger$ and χ_{s_2}. We call I the total isospin of the two nucleons, T the total isospin of the pion q and the nucleon q_1. The P_I's are the projection operators for isospin I, and the numerical 2×2 matrix α_{IT} has been defined in Appendix 1, Eq. (A12). Finally, $J^{(T)}$ is given by

$$\begin{aligned}J^{(T)}=\int_\Omega\frac{d^3q_1\,d^3q}{q_{10}q_0}\delta^4(p_1+p_2-q_1-q_2-q)\Big[&(f_1{}^{(T)\prime})^*f_1{}^{(T)}\\&+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{p}_1}{p_1'p_1}(f_2{}^{(T)\prime})^*f_2{}^{(T)}+\frac{\boldsymbol{\sigma}\cdot\mathbf{q}_1\boldsymbol{\sigma}\cdot\mathbf{p}_1}{q_1p_1}(f_1{}^{(T)\prime})^*f_2{}^{(T)}\\&+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{q}_1}{p_1'q_1}(f_2{}^{(T)\prime})^*f_1{}^{(T)}\Big]^{(1)},\end{aligned}\quad(34)$$

[6] G. Da Prato, Nuovo cimento **22**, 123 (1961).

[7] E. Ferrari and F. Selleri, Nuovo cimento **21**, 1028 (1961).

where the $f^{(T)}$'s are the transition amplitudes for off-shell πN scattering at fixed isospin T. We now remark that $J^{(T)}$ is exactly the integral which one has to calculate in the right-hand side of the unitarity condition for πN scattering below the threshold for inelastic processes. The only difference lies in the definition of the partial-wave amplitudes

$$f_1^{(T)}=\sum_{l=0}^{\infty} f_{l+}^{(T)}(\omega^2,\Delta^2)P_{l+1}'(\cos\delta)-\sum_{l=2}^{\infty} f_{l-}^{(T)}(\omega^2,\Delta^2)P_{l-1}'(\cos\delta), \quad (35)$$

$$f_2^{(T)}=\sum_{l=1}^{\infty}\left[f_{l-}^{(T)}(\omega^2,\Delta^2)-f_{l+}^{(T)}(\omega^2,\Delta^2)\right]P_l'(\cos\delta),$$

and similarly for $f_1^{(T)\prime}$ and $f_2^{(T)\prime}$, which now depend on Δ^2 (Δ'^2 for the primed case). We have called δ the scattering angle from $\mathbf{q}_1$ to $\mathbf{p}_1$ in the system $\mathbf{q}+\mathbf{q}_1=0$.

If we had no dependence on Δ^2 and Δ'^2 the calculation of Eq. (34) would give

$$J^{(T)}=\frac{4\pi q(\omega)}{\omega}\left[g_1^{(T)}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{p}_1}{p_1'p_1}g_2^{(T)}\right], \quad (36)$$

where $q(\omega)$ is the modulus of the 3-momentum in the system $\mathbf{q}+\mathbf{q}_1=0$, and g_1 and g_2 would be exactly of the form (35) with each partial-wave amplitude substituted with its squared modulus. This is so because on the partial-wave amplitudes the unitarity condition reads

$$\mathrm{Im} f_{l\pm}=q|f_{l\pm}|^2,$$

and the left-hand side of the unitarity condition for πN scattering is of the form (36) with $\mathrm{Im} f_1\to q(\omega)g_1$ and $\mathrm{Im} f_2\to q(\omega)g_2$.

In our case the result (36) still holds, because Δ^2 and Δ'^2 appear only in the partial-wave amplitudes. We have, however, a little change, namely, the replacement of $|f_{l\pm}|^2$ with $(f_{l\pm}')^*f_{l\pm}$. More explicitly the result is

$$g_1^{(T)}=\sum_{l=0}^{\infty}(f_{l+}^{(T)\prime})^*f_{l-}^{(T)}P_{l+1}'(\cos\xi)-\sum_{l=2}^{\infty}(f_{l-}^{(T)\prime})^*f_{l-}^{(T)}P_{l-1}'(\cos\xi), \quad (37)$$

$$g_2^{(T)}=\sum_{l=1}^{\infty}\left[(f_{l-}^{(T)\prime})^*f_{l-}^{(T)}-(f_{l+}^{(T)\prime})^*f_{l+}^{(T)}\right]P_l'(\cos\xi),$$

where ξ is the scattering angle from $\mathbf{p}_1$ to $\mathbf{p}_1'$ in the system $\mathbf{q}+\mathbf{q}_1=0$.

The above simple argument can be checked with a direct but lengthy calculation, by using the properties of the Legendre polynomials reported in Appendix 2.

We have so carried on the calculation as far as possible in the system $\mathbf{q}+\mathbf{q}_1=0$. The next integrations to be performed are on the vector $\mathbf{q}_2$ and are most simply done in the over-all c.m. system. In order to make the needed transformation, we write $\chi_{s_1'}{}^\dagger J^{(T)}\chi_{s_1}$ in the invariant form

$$\chi_{s_1'}{}^\dagger J^{(T)}\chi_{s_1}=\frac{mq(\omega)}{\omega^2}\times\bar{u}(p_1')\left[-C^{(T)}+i\gamma\cdot\Delta D^{(T)}\right]u(p_1), \quad (38)$$

where

$$C^{(T)}=-4\pi\left[\frac{(\omega+m)g_1^{(T)}}{[(p_{10}+m)(p_{10}'+m)]^{1/2}}-\frac{(\omega-m)g_2^{(T)}}{[(p_{10}-m)(p_{10}'-m)]^{1/2}}\right], \quad (39)$$

$$D^{(T)}=-4\pi\left[\frac{g_1^{(T)}}{[(p_{10}+m)(p_{10}'+m)]^{1/2}}+\frac{g_2^{(T)}}{[(p_{10}-m)(p_{10}'-m)]^{1/2}}\right], \quad (40)$$

with

$$p_{10}=(\omega^2+m^2+\Delta^2)/2\omega;\quad p_{10}'=(\omega^2+m^2+\Delta'^2)/2\omega. \quad (41)$$

Transforming Eq. (38) back to bi-dimensional spinsors, this time in the over-all c.m. system, we get

$$\begin{aligned}R_{11}=&\frac{f^2}{2\pi^2}\frac{\frac{1}{2}W+m}{\mu^2}\int_{\Omega/2}\frac{d^3q_2}{q_{20}}(q_{20}+m)q(\omega)P(\Delta^2)P(\Delta'^2)\\&\times\chi_{s_2'}{}^\dagger\chi_{s_1'}{}^\dagger\left\{\frac{q_{20}-m}{q_{20}+m}+\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{p}_1}{(\frac{1}{2}W+m)^2}+\frac{1}{(q_{20}+m)(\frac{1}{2}W+m)}\left[\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{q}_2+\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{p}_1\right]\right\}^{(2)}\\&\times\sum_{IT}P_{I\alpha_{IT}}\left\{(\tfrac{1}{2}W+m)\left[C^{(T)}+(W-m-q_{20})D^{(T)}\right]-\left[C^{(T)}-(W+m-q_{20})D^{(T)}\right]\frac{\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{p}_1}{\frac{1}{2}W+m}\right.\\&\left.+D^{(T)}\left[\boldsymbol{\sigma}\cdot\mathbf{p}_1'\boldsymbol{\sigma}\cdot\mathbf{q}_2+\boldsymbol{\sigma}\cdot\mathbf{q}_2\boldsymbol{\sigma}\cdot\mathbf{p}_1\right]\right\}^{(1)}\chi_{s_2}\chi_{s_1}. \quad (42)\end{aligned}$$

In the over-all c.m. system the invariants Δ^2 and Δ'^2 reduce to

$$\begin{aligned}\Delta^2&=-2m^2-[W^2-4m^2]^{1/2}q_2\cos\bar{\vartheta}+Wq_{20},\\ \Delta'^2&=-2m^2-[W^2-4m^2]^{1/2}q_2\cos\bar{\vartheta}'+Wq_{20}.\end{aligned} \quad (43)$$

$\bar{\vartheta}$ and $\bar{\vartheta}'$ are represented in Fig. 4, where also ϑ, the c.m. scattering angle for the elastic process, has been indicated.

In order to obtain an expression analogous to Eq. (10) for R_{11}, we must express the vectors appearing in

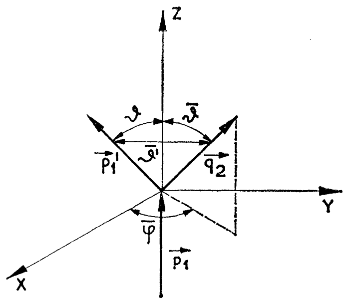

FIG. 4. Sketch of the angles appearing in Eq. (43).

Eq. (42) as functions of the three fundamental vectors **N**, **P**, and **K**, which have been defined in Eq. (11). The result of such a manipulation is

$$R_{11}=\chi_{s_2'}{}^\dagger\chi_{s_1'}{}^\dagger\int_{\Omega/2}\frac{d^3q_2}{q_{20}}H_0[H_1H_2+i\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}H_1H_3+i\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}H_2H_4-\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}H_3H_4+\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P}2(1-\cos\vartheta)H_5H_6]\chi_{s_2}\chi_{s_1},\quad(44)$$

where

$$H_0=-\frac{f^2}{2\pi^2}\frac{\frac{1}{2}W+m}{\mu^2}(q_{20}+m)q(\omega)P(\Delta^2)P(\Delta'^2)\times\sum_{IT}P_I\alpha_{IT},\quad(45)$$

$$H_1=\frac{q_{20}-m}{q_{20}+m}+\frac{\frac{1}{2}W-m}{\frac{1}{2}W+m}\cos\vartheta-\lambda(\cos\bar{\vartheta}+\cos\bar{\vartheta}'),\quad(46)$$

$$H_2=\cos\vartheta(\tfrac{1}{2}W-m)[C^{(T)}-(W+m-q_{20})D^{(T)}]-(\tfrac{1}{2}W+m)[C^{(T)}+(W-m-q_{20})D^{(T)}]+\nu(\cos\bar{\vartheta}+\cos\bar{\vartheta}')D^{(T)},\quad(47)$$

$$H_3=-\sin\vartheta(\tfrac{1}{2}W-m)[C^{(T)}-(W+m-q_{20})D^{(T)}]-\nu[\sin\bar{\vartheta}\cos\bar{\varphi}(1-\cos\vartheta)+\cos\bar{\vartheta}\sin\vartheta]D^{(T)},\quad(48)$$

$$H_4=-\sin\vartheta\frac{\frac{1}{2}W-m}{\frac{1}{2}W+m}+\lambda[\sin\bar{\vartheta}\cos\bar{\varphi}(1-\cos\vartheta)+\cos\bar{\vartheta}\sin\vartheta],\quad(49)$$

$$H_5=\lambda\sin\bar{\vartheta}\sin\bar{\varphi},\quad(50)$$

$$H_6=\nu\sin\bar{\vartheta}\sin\bar{\varphi}\,D^{(T)},\quad(51)$$

$$\lambda=\left[\frac{(q_{20}-m)(\frac{1}{2}W-m)}{(q_{20}+m)(\frac{1}{2}W+m)}\right]^{1/2};\quad\nu=[(q_{20}{}^2-m^2)(\tfrac{1}{4}W^2-m^2)]^{1/2}.\quad(52)$$

Finally, we must symmetrize this result according to the prescriptions of Eq. (26) in which R_{12} is neglected. In the notations of Sec. III we have now calculated the term $R_{11}(p',p)$. The first term to be added is $R_{11}(\bar{p}',\bar{p})$, where $\bar{p}'(\bar{p})$ means exchanging $p_1'\leftrightarrow p_2'$ and $s_1'\leftrightarrow s_2'$ ($p_1\leftrightarrow p_2$ and $s_1\leftrightarrow s_2$). While the exchanges $p_1'\leftrightarrow p_2'$ and $p_1\leftrightarrow p_2$, when performed together, do not affect the W and $\cos\vartheta$ dependence of Eq. (44), the exchanges $s_1'\leftrightarrow s_2'$ *and* $s_1\leftrightarrow s_2$ are fully equivalent in the two-dimensional notation, to the exchange $\boldsymbol{\sigma}^{(1)}\leftrightarrow\boldsymbol{\sigma}^{(2)}$.

In fact the label (1) [(2)] on the spin operators means only that the matrix element of such an operator has to be taken between the spinors $\chi_{s_1'}{}^\dagger$ and χ_{s_1} ($\chi_{s_2'}{}^\dagger$ and χ_{s_2}). Therefore,

$$2R_{11}(p',p)+2R_{11}(\bar{p}',\bar{p})=\chi_{s_2'}{}^\dagger\chi_{s_1'}{}^\dagger\int_{\Omega/2}\frac{d^3q_2}{q_{20}}H_0[4H_1H_2+i(\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}+\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N})\times2(H_1H_3+H_2H_4)-\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N}4H_3H_4+\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P}8(1-\cos\vartheta)H_5H_6]\chi_{s_2}\chi_{s_1}.\quad(53)$$

Similarly, by using the obvious property

$$R_{11}(\bar{p}',p)=R_{11}(\bar{p}',\bar{p}),$$

one obtains the sum of the third and fourth term in Eq. (26) simply by exchanging the initial particles in the previous relation. One thus gets

$$2R_{11}(p',\bar{p})+2R_{11}(\bar{p}',p)=\chi_{s_2'}{}^\dagger\chi_{s_1'}{}^\dagger\int_{\Omega/2}\frac{d^3q_2}{q_{20}}\bar{H}_0[4\bar{H}_1\bar{H}_2-i(\boldsymbol{\sigma}^{(12)}\cdot\mathbf{N}+\boldsymbol{\sigma}^{(21)}\cdot\mathbf{N})\times2(\bar{H}_1\bar{H}_3+\bar{H}_2\bar{H}_4)-\boldsymbol{\sigma}^{(12)}\cdot\mathbf{N}\boldsymbol{\sigma}^{(21)}\cdot\mathbf{N}4\bar{H}_3\bar{H}_4+\boldsymbol{\sigma}^{(12)}\cdot\mathbf{K}\boldsymbol{\sigma}^{(21)}\cdot\mathbf{K}8(1+\cos\vartheta)\bar{H}_5\bar{H}_6]\chi_{s_1}\chi_{s_2},\quad(54)$$

where the bar over an amplitude means that the exchange $\cos\vartheta\to-\cos\vartheta$ has been performed. In the case of H_0 the bar indicates also the exchange of the isospin zero projection operator into minus itself: $P_0\to-P_0$. Furthermore, in deducing Eq. (54) we have taken into account that the exchange $p_1\leftrightarrow p_2$ gives $\mathbf{N}\to-\mathbf{N}$ and $\mathbf{P}\to-\mathbf{K}$. As before, the labels on the spin operators indicate between which spinor states their matrix element, e.g., must be taken

$$\chi_{s_2'}{}^\dagger\chi_{s_1'}{}^\dagger\boldsymbol{\sigma}^{(12)}\cdot\mathbf{N}\chi_{s_1}\chi_{s_2}=\chi_{s_2'}{}^\dagger\chi_{s_1}\cdot\chi_{s_1'}{}^\dagger\boldsymbol{\sigma}\cdot\mathbf{N}\chi_{s_2}.$$

In order to reduce to a unique form the difference of Eqs. (53) and (54), the next problem is to express the matrix elements in the "new" ordering (54) to matrix elements of the "old" ordering (53). This problem is analogous to the problem of Fierz transformation in the 4-dimensional case. By defining

$$\begin{aligned}a_{\mathrm{op}}&=\mathbf{1}^{(1)}\mathbf{1}^{(2)},\\c_{\mathrm{op}}&=\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}\mathbf{1}^{(2)}+\mathbf{1}^{(1)}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N},\\m_{\mathrm{op}}&=\boldsymbol{\sigma}^{(1)}\cdot\mathbf{N}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{N},\\d_{\mathrm{op}}&=\boldsymbol{\sigma}^{(1)}\cdot\mathbf{P}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{P},\\f_{\mathrm{op}}&=\boldsymbol{\sigma}^{(1)}\cdot\mathbf{K}\boldsymbol{\sigma}^{(2)}\cdot\mathbf{K},\end{aligned}\quad(55)$$

and $\bar{a}_{op}=\mathbf{1}^{(12)}\mathbf{1}^{(21)}$, and so on, the transformation from the barred to the unbarred operators is given by

$$\begin{bmatrix}\bar{a}_{op}\\ \bar{c}_{op}\\ \bar{m}_{op}\\ \bar{d}_{op}\\ \bar{f}_{op}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&0&1&1&1\\0&2&0&0&0\\1&0&1&-1&-1\\1&0&-1&1&-1\\1&0&-1&-1&1\end{bmatrix}\begin{bmatrix}a_{op}\\ c_{op}\\ m_{op}\\ d_{op}\\ f_{op}\end{bmatrix}. \qquad (56)$$

Finally, using Eqs. (54), (56) and (53) we can write R_A in the form (10), namely,

$$R_A=\chi_{s_2'}{}^{\dagger}\chi_{s_1'}{}^{\dagger}[2a_0a_{op}+2c_0c_{op}+2m_0m_{op}+2d_0d_{op}+2f_0f_{op}]\chi_{s_2}\chi_{s_1}. \qquad (57)$$

As we have already pointed out, the factor 2 is put in for convenience. The amplitudes $a_0, \cdots f_0$ are given by

$$\begin{aligned}
a_0&=\int_{\Omega/2}\frac{d^3q_2}{q_{20}}[2H_0H_1H_2-\bar{H}_0\bar{H}_1\bar{H}_2+\bar{H}_0\bar{H}_3\bar{H}_4-2(1+\cos\vartheta)\bar{H}_0\bar{H}_5\bar{H}_6],\\
c_0&=\int_{\Omega/2}\frac{d^3q_2}{q_{20}}[H_0(H_1H_3+H_2H_4)+\bar{H}_0(\bar{H}_1\bar{H}_3+\bar{H}_2\bar{H}_4)],\\
m_0&=\int_{\Omega/2}\frac{d^3q_2}{q_{20}}[-2H_0H_3H_4-\bar{H}_0\bar{H}_1\bar{H}_2+\bar{H}_0\bar{H}_3\bar{H}_4+2(1+\cos\vartheta)\bar{H}_0\bar{H}_5\bar{H}_6],\\
d_0&=\int_{\Omega/2}\frac{d^3q_2}{q_{20}}[4(1-\cos\vartheta)H_0H_5H_6-\bar{H}_0\bar{H}_1\bar{H}_2-\bar{H}_0\bar{H}_3\bar{H}_4+2(1+\cos\vartheta)\bar{H}_0\bar{H}_5\bar{H}_6],\\
f_0&=\int_{\Omega/2}\frac{d^3q_2}{q_{20}}[-\bar{H}_0\bar{H}_1\bar{H}_2-\bar{H}_0\bar{H}_3\bar{H}_4-2(1+\cos\vartheta)\bar{H}_0\bar{H}_5\bar{H}_6].
\end{aligned} \qquad (58)$$

Notwithstanding the fact that the full amplitude is symmetric under the exchange of the initial or final particles, the amplitudes (58) do not satisfy any simple symmetry property (with the exception of c_0). This is due to the fact that the spin factors with which they are multiplied in Eq. (10) do not possess symmetry properties either.

Apart from numerical integrations, Eqs. (58) solve the problem of calculating the contribution of the one-pion–two-nucleon inelastic channel to the unitarity condition.

V. NUMERICAL RESULTS AND CONCLUSIONS

In this section we give the results of a numerical calculation of the amplitudes (58) at two energies, 0.97 and 1.2 GeV. In the diffraction approximation we also discuss the solution of the Eqs. (13) which gives us the scalar amplitudes $a, \cdots f$ of the M-matrix element for N-N scattering in terms of $a_0, \cdots f_0$. In the triple integrations appearing in (58) we can change variables in the following way:

$$d^3q_2/q_{20}=(q_2\omega/W)d\omega d\cos\bar{\vartheta}d\bar{\varphi}, \qquad (59)$$

where ω is defined in Eq. (30), and $\bar{\vartheta}$, $\bar{\varphi}$ in Fig. 4. It should be noted that some of the amplitudes $H_0, \cdots H_6$ are also functions of $\cos\bar{\vartheta}'$, which is given by

$$\cos\bar{\vartheta}'=\cos\vartheta\cos\bar{\vartheta}+\sin\vartheta\sin\bar{\vartheta}\cos\bar{\varphi}. \qquad (60)$$

Since the integrations in (58) are over half of the allowed phase space the limits are

$$\begin{gathered}0\leq\bar{\varphi}\leq\pi,\\ -1\leq\cos\bar{\vartheta}\leq+1,\\ m+\mu\leq\omega\leq W-m.\end{gathered} \qquad (61)$$

For the off-shell πN scattering we have neglected all the partial waves, except the $T=J=3/2$ one, for which the expression given in reference 2 has been assumed. As it was shown in that paper, this is a very good approximation for lab kinetic energies between 0.8 GeV and about 1.3 GeV. This is another reason for which our theory can be valid at most in this energy range.

From the structure of the matrix α_{IT} [see Eq. (A12)] it results that only $T=1$ diffraction scattering is predicted in this approximation.

By using the CERN Ferranti computer we have calculated the amplitudes (58) at lab kinetic energies of 0.97 and 1.2 GeV. The results are reported in Fig. 5 as solid lines in units (nucleon mass)$^{-2}$ ($1m^{-2}=0.442$ mb). The precision of the calculated curves due to finite step lengths is estimated to be greater than 5%. Since forwards one is essentially calculating the total cross section for single-pion production (σ_{in}), one must have

$$\sigma_{in}=\frac{m^2}{4W}\frac{1}{(\frac{1}{4}W^2-m^2)^{1/2}}\sum_{s_2s_1}R_A(\vartheta=0;s_2'=s_2;s_1'=s_1)=\frac{2m^2}{W}\frac{a_0(0)}{(\frac{1}{4}W^2-m^2)^{1/2}}. \qquad (62)$$

Indeed, for example, at 0.97 GeV we get $\sigma_{in}=23.1$ mb, while experimentally $\sigma_{in}=(22.7\pm1.2)$ mb.

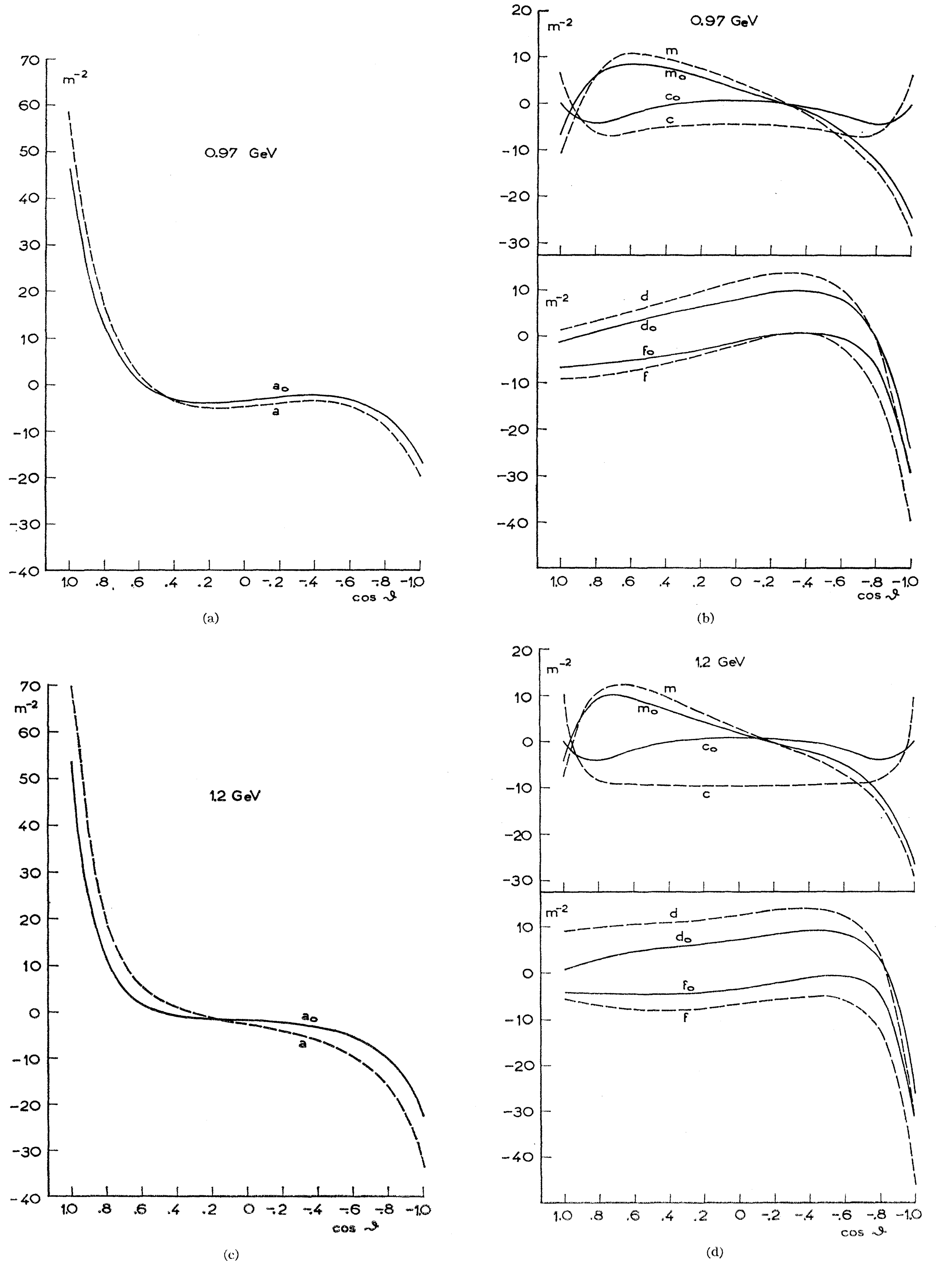

FIG. 5. Full lines—scalar amplitudes contributed to the unitarity condition for nucleon-nucleon scattering by the one-pion–two-nucleon inelastic channel. Dashed lines—scalar amplitudes obtained by solving the unitarity conditions in the diffraction approximation.

The next step consists in introducing the obtained $a_0, \cdots f_0$ in Eq. (13) and in solving these equations for $a, \cdots f$. In the diffraction approximation, in which all the amplitudes are purely imaginary, we can forget the Im in the left-hand side of Eq. (13).

Equations (13) together with Eqs. (18), (19), and (20) now constitute a set of five nonlinear integral equations for which a general method of solution does not exist. One possibility is to introduce the parameter

$$\epsilon=\frac{1}{(2\pi)^2}\frac{P}{4W}, \tag{63}$$

and to write each unknown dimensionless amplitude $m^2a, \cdots m^2f=u^{(1)}, \cdots u^{(5)}$ as a power series in ϵ:

$$u^{(i)}=u_0^{(i)}+\epsilon u_1^{(i)}+\epsilon^2u_2^{(i)}+\cdots. \tag{64}$$

Substituting Eq. (64) in Eq. (13) one gets for the successive terms of the amplitudes equations of the type

$$u_{s+1}^{(i)}=\int_\Omega d\Omega \sum_{j,k=1}^{5}\sum_{n=0}^{s} c(i,j,k)u_{s-n}^{(j)\prime\prime}u_n^{(k)\prime}; \qquad (i=1\cdots 5) \tag{65}$$

where the coefficients $c(i,j,k)$ can be read out of formulas (18), (19), and (20). Equation (65) expresses the fact that each term of the power series can be calculated knowing the lower ones. The values of the ϵ parameter are 1.85×10^{-3} and 1.98×10^{-3} at 0.97 and 1.2 GeV, respectively. The results of such a numerical calculation with $s_{\max}=5$ are shown in Fig. 5 as dashed curves.

The convergence of the series at 0.97 GeV is rather good, while it is less favorable at 1.2 GeV. It is quite clear that we need a better solution of the system (13). Such a solution could be obtained, for instance, by developing each scalar amplitude $u^{(i)}$ in series of Legendre polynomials. Thus equations (13) become a set of five quadratic algebraic equations which can be solved exactly.

The differential cross section given by

$$d\sigma/d\Omega=[m^4/(2\pi)^2W^2] \times[|a|^2+|m|^2+2|c|^2+|d|^2+|f|^2] \tag{66}$$

is shown in Fig. 6 as a dashed line together with the experimental data for pp scattering at the same energies. As we have already remarked, Eq. (66) is a cross section for the $I=1$ state, while that for $I=0$ is predicted to be zero. This means that with our approximations we predict that the differential cross section for neutron-proton scattering is at every angle one fourth of that for proton-proton scattering. As a consequence, the neutron-proton scattering cross section should be symmetric around 90° and therefore also show a backward peak. This is in agreement with experimental observation.[8] In

[8] A. P. Batson, B. B. Culwick, H. B. Klepp, and L. Riddiford, Proc. Roy. Soc. (London) **A251**, 233 (1959).

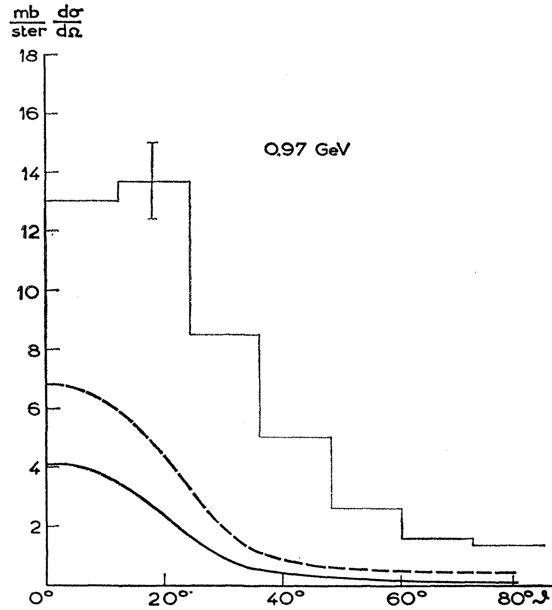

FIG. 6. Experimental data for proton-proton scattering at 0.97 GeV compared with the differential cross section obtained by solving the unitarity conditions in the diffraction approximation (dashed line). The full line represents the cross section obtained using the amplitudes $a_0\cdots f_0$ drawn as full lines in Fig. 5(a) and 5(b).

Fig. 6 the continuous curve represents the same cross section as calculated with Eq. (66) using, however, the amplitudes $a_0, \cdots f_0$. It is seen that the iterative procedure with which we passed from $a_0, \cdots f_0$ to $a, \cdots f$ does not increase enough the differential cross section to obtain agreement with the experimental data. Since our calculation under the assumption that the real parts of the phase shifts are equal to zero is a rigorous one, we must conclude that this assumption is incorrect and that potential scattering cannot be neglected at these energies. It can easily be verified that an $\alpha_\rho\neq0$ can indeed increase the cross section. One has for every partial-wave amplitude f_ρ

$$|f_\rho|^2=\tfrac{1}{4}[1+e^{-4\beta_\rho}-2e^{-2\beta_\rho}\cos2\alpha_\rho], \tag{67}$$

which, fixed β_ρ, has its minimum value for $\alpha_\rho=0$. On the other hand, the potential we need must also satisfy the condition of giving a vanishing contribution at zero momentum transfer, since the optical theorem alone is able to reproduce the experimental forward cross section. The most natural potential satisfying such a condition is the one-pion-exchange term in the elastic channel. It is our feeling that with the unitarity condition alone it is not possible to go further, and that dispersion relations are needed in order to introduce this term in a self-consistent way in the theory. The discussion of such a problem is left for a future paper.

We wish to thank Professor S. Fubini for an illuminating discussion. We are also indebted to Professor D. Amati and Professor A. Stanghellini for useful

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Note added in proof. A wrong numerical coefficient in Eq. (13) was contained in the first version of this paper. The error affected practically only the c amplitude, which is now much closer to zero than it was before. The cross section as given by the dashed line in Fig. 6, however, is changed only by about 5%, thus leaving our conclusions unchanged.

APPENDIX 1

In this Appendix we wish to perform the summations on the charge indices α, β, and γ which appear in Eqs. (23) and (31) of the text.

The quantities we have to evaluate are of the form

$$g_{ij}=\sum_{\alpha\beta\gamma}\langle\tau^\beta\tau^\gamma\rangle\langle(f_i^{\alpha\beta\prime})^*f_j^{\alpha\gamma}\rangle, \tag{A1}$$

where the f's are amplitudes with the structure

$$f_i^{\alpha\beta}=\delta_{\alpha\beta}f_i^{(+)}+\tfrac{1}{2}[\tau^\alpha,\tau^\beta]f_i^{(-)}, \tag{A2}$$

the $f_i^{(\pm)}$ being further related to the fixed isospin amplitudes by the well-known relations

$$\begin{aligned}f_i^{(+)}&=\tfrac{1}{3}(f_i^{(1/2)}+2f_i^{(3/2)}),\\ f_i^{(-)}&=\tfrac{1}{3}(f_i^{(1/2)}-f_i^{(3/2)}).\end{aligned} \tag{A3}$$

On substitution of the expressions for $(f_i^{\alpha\beta\prime})^*$ and $f_j^{\alpha\gamma}$ as given in Eq. (A2) one gets

$$\begin{aligned}g_{ij}=\sum_{\beta\gamma}\langle\tau^\beta\tau^\gamma\rangle\langle\delta_{\beta\gamma}[(f_i^{(+)\prime})^*f_j^{(+)}+2(f_i^{(-)\prime})^*f_j^{(-)}]\\ +\tfrac{1}{2}[\tau^\beta,\tau^\gamma][(f_i^{(+)\prime})^*f_j^{(-)}+(f_i^{(-)\prime})^*f_j^{(+)}\\ +(f_i^{(-)\prime})^*f_j^{(-)}]\rangle.\end{aligned} \tag{A4}$$

By writing $\tau^\beta\tau^\gamma=\delta_{\beta\gamma}+\frac{1}{2}[\tau^\beta,\tau^\gamma]$ in the first bracket and substituting $\frac{1}{2}[\tau^\beta,\tau^\gamma]=i\epsilon_{\beta\gamma\lambda}\tau^\lambda$ everywhere, the β and γ summations can easily be carried out. The result is

$$g_{ij}=\langle\langle P_0G_{ij}^0+P_1G_{ij}^1\rangle\rangle, \tag{A5}$$

where

$$P_0=\tfrac{1}{4}(1-\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}),\quad P_1=\tfrac{1}{4}(3+\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}) \tag{A6}$$

are the usual isospin projection operators for the nucleon-nucleon system and

$$\begin{aligned}G_{ij}^0=3[(f_i^{(+)\prime})^*f_j^{(+)}+2(f_i^{(+)\prime})^*f_j^{(-)}\\ +2(f_i^{(-)\prime})^*f_j^{(+)}+4(f_i^{(-)\prime})^*f_j^{(-)}],\end{aligned} \tag{A7}$$

$$\begin{aligned}G_{ij}^1=3(f_i^{(+)\prime})^*f_j^{(+)}-2(f_i^{(+)\prime})^*f_j^{(-)}\\ -2(f_i^{(-)\prime})^*f_j^{(+)}+4(f_i^{(-)\prime})^*f_j^{(-)}.\end{aligned} \tag{A8}$$

These expressions for G_{ij}^0 and G_{ij}^1 can be further transformed if one uses the relations (A3) for $(f_i^{(\pm)\prime})^*$ and $f_j^{(\pm)}$. So one gets

$$G_{ij}^0=3(f_i^{(1/2)\prime})^*f_j^{(1/2)}, \tag{A9}$$

$$G_{ij}^1=\tfrac{1}{3}(f_i^{(1/2)\prime})^*f_j^{(1/2)}+8/3(f_i^{(3/2)\prime})^*f_j^{(3/2)}. \tag{A10}$$

By inserting Eqs. (A9) and (A10) into Eq. (A5) one easily recognizes that the resulting expression can be written

$$g_{ij}=\langle\langle\sum_{IT}P_I\alpha_{IT}(f_i^{(T)\prime})^*f_j^{(T)}\rangle\rangle, \tag{A11}$$

where $I=0, 1$ is the total isospin of the NN system, $T=1/2, 3/2$ is the total isospin of the intermediate πN system and the matrix α_{IT} is given by

$$\alpha_{IT}=\begin{pmatrix}3 & 0\\ 1/3 & 8/3\end{pmatrix}. \tag{A12}$$

The fact that $\alpha_{0\frac{3}{2}}=0$ is easy to understand, since a πN system with $T=3/2$ cannot give $I=0$ by summing its isospin to that of the remaining nucleon.

APPENDIX 2

Some Integral Properties of Legendre Polynomials and Their First Derivatives

The starting point is the well-known rule for the sum of two Legendre polynomials on the sphere which, using the angles as defined in Fig. 4, can be written

$$\begin{aligned}P_n(\cos\bar{\vartheta}')=P_n(\cos\vartheta)P_n(\cos\bar{\vartheta})\\ +2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\bar{\vartheta})P_n^m(\cos\vartheta)\cos m\bar{\varphi}.\end{aligned} \tag{A13}$$

We also recall the orthonormality condition

$$\int_{-1}^{+1}P_n(x)P_l(x)dx=\frac{2}{(2n+1)^{1/2}}\frac{1}{(2l+1)^{1/2}}\delta_{nl}. \tag{A14}$$

By multiplying Eq. (A13) by $P_l(\cos\bar{\vartheta})$ and by integrating on Ω, $d\bar{\Omega}=d\cos\bar{\vartheta}d\bar{\varphi}$ one gets the well-known formulas

$$\int d\bar{\Omega}\,P_n(\cos\bar{\vartheta}')P_l(\cos\bar{\vartheta})=\frac{4\pi}{2n+1}\delta_{nl}P_l(\cos\vartheta). \tag{A15}$$

By differentiation of Eq. (A13) with respect to $\bar{\varphi}$, subsequent multiplication by $P_l'(\cos\bar{\vartheta})$ and $P_l(\cos\bar{\vartheta})$ and integration, we get

$$\begin{aligned}\int d\bar{\varphi}\,\sin\bar{\vartheta}\,\sin\bar{\varphi}P_n'(\cos\bar{\vartheta}')P_l'(\cos\bar{\vartheta})=0,\\ \int d\bar{\Omega}\,\sin\bar{\vartheta}\,\sin\bar{\varphi}P_n'(\cos\bar{\vartheta}')P_l(\cos\bar{\vartheta})=0.\end{aligned} \tag{A16}$$

The first derivative of a Legendre polynomial can be written as a sum of polynomials of lower order according to the expansion

$$\begin{aligned}P_n'=(2n-1)P_{n-1}+(2n-5)P_{n-3}\\ +(2n-9)P_{n-5}+\cdots.\end{aligned} \tag{A17}$$

Using Eqs. (A17) and (A15) it is easy to obtain the relation

$$\int d\bar{\Omega}\, P_n'(\cos\bar{\vartheta}')P_l'(\cos\bar{\vartheta}) = 4\pi P_{(n;l)}'(\cos\vartheta)\Delta_{|l+1-n|}, \quad \text{(A18)}$$

where we have introduced the symbols $(n;l)$ and Δ_r with the meanings:

$$(n;l) \equiv \text{the smaller of the two indices } n \text{ and } l;$$
$$\Delta_r = 1 \text{ if } r>0 \text{ and odd}, \quad \text{(A19)}$$
$$= 0 \text{ if } r\leq 0 \text{ or } r>0 \text{ even}.$$

By differentiation of Eq. (A13) with respect to the different variables which it contains, and by proper multiplication by Legendre polynomials and their derivatives and integration, one gets other useful equations

$$\int d\bar{\Omega}\, P_n(\cos\bar{\vartheta}')P_l'(\cos\bar{\vartheta}) = 4\pi P_n(\cos\vartheta)\Delta_{(l-n)}, \quad \text{(A20)}$$

$$\int d\bar{\Omega}\, \sin\bar{\vartheta}\, \cos\bar{\varphi} P_n'(\cos\bar{\vartheta}')P_l'(\cos\bar{\vartheta}) = 4\pi \sin\vartheta[P_{(l;n+1)}'(\cos\vartheta)\Delta_{|l-n|} - P_{n+1}'(\cos\vartheta)\Delta_{(l-n)}], \quad \text{(A21)}$$

$$\int d\bar{\Omega}\, \cos\bar{\vartheta} P_n'(\cos\bar{\vartheta}')P_l'(\cos\bar{\vartheta}) = 4\pi[\cos\vartheta P_{(l;n+1)}'(\cos\vartheta)\Delta_{|l-n|} - (n+1)P_{n+1}(\cos\vartheta)\Delta_{(l-n)}]. \quad \text{(A22)}$$

The last one also gives the result for $\cos\bar{\vartheta}$ replaced by $\cos\bar{\vartheta}'$ by interchanging $l\leftrightarrow n$. Finally from Eq. (A21) and the equation

$$(2l+1)P_l = P_{l+1}' - P_{l-1}', \quad \text{(A23)}$$

one gets

$$\int d\bar{\Omega}\, \sin\bar{\vartheta}\, \cos\bar{\varphi} P_n(\cos\bar{\vartheta}')P_l(\cos\bar{\vartheta}) = \frac{4\pi \sin\vartheta}{(2l+1)(2n+1)} P_n'(\cos\vartheta)(\delta_{n,l+1} - \delta_{n,l-1}), \quad \text{(A24)}$$

while Eqs. (A22) and (A23) imply

$$\int d\bar{\Omega}\, \cos\bar{\vartheta} P_n(\cos\bar{\vartheta}')P_l(\cos\bar{\vartheta}) = \frac{4\pi}{(2l+1)(2n+1)} P_n(\cos\vartheta) \times[(n+1)\delta_{n,l-1} + n\delta_{n,l+1}]. \quad \text{(A25)}$$