Higher Born Approximations for the Coulomb Scattering of a Spinless Particle*

IK-JU KANG† AND LAURIE M. BROWN
Northwestern University, Evanston, Illinois
(Received July 5, 1962)

The Coulomb scattering amplitude of a Klein-Gordon particle is computed exactly to the third order in $\beta=Ze^2/\hbar v$ by perturbation theory using the potential $Zee^{-\lambda r}/r$ in the limit $\lambda\rightarrow 0$. It contains only finite terms except for terms containing powers of $\ln(|\mathbf{Q}|/\lambda)$ which can be shown to arise from the expansion of a common phase factor. Thus, first- and second-order corrections to the relativistic Rutherford cross section are obtained.

Our results are compared to the exact scattering amplitude in the form of the partial-wave summation, expanded in powers of β, which we have succeeded in summing to third order.

I. INTRODUCTION

It is a well-known fact that the exact Coulomb scattering amplitude for a nonrelativistic spinless particle and the first term of the corresponding Born approximation series differ only by a phase factor, whose argument, however, is, strictly speaking, infinite. Since the succeeding terms of the Born expansion contain terms which become infinite for the unscreened potential, Dalitz[1] has conjectured that these terms represent merely the expansion of the phase factor multiplying the lowest order term. He succeeded, in fact, in verifying this conjecture exactly to second order and approximately in third order for the Schrödinger particle. Later, Kacser[2] confirmed his result, calculating the third order exactly. Dalitz also considered the Dirac particle, obtaining finite relativistic corrections to the second order, as well as terms containing $\ln\lambda$, where λ^{-1} is the screening radius, which terms he again interpreted as the expansion of a phase factor.

In this paper our intent is to show that Dalitz's conjecture holds also for the relativistic spinless particle by showing that to the third order in $\beta=Ze^2/\hbar v$, the Born series can be grouped in the form of a product of an infinitely large (for zero screening) phase factor and a finite correction factor, and obtaining, thus, corrections to the relativistic Rutherford scattering cross section.

We have also made use of a second independent approach to this problem, namely, the scattering amplitude for the Klein-Gordon particle expressed as a partial-wave sum. We use the fact that two forms of the nonrelativistic Coulomb scattering amplitude are known, namely, that in closed form, obtained by separation in parabolic coordinates, as well as the partial-wave sum obtained from the separation in spherical coordinates. Expanding each of these forms in powers of β, comparison yields some of the terms which appear in the scattering amplitude for the Klein-Gordon particle. To third order, one is able to verify, in agreement with the result of the perturbation calculation, that the Born series for the Klein-Gordon particle is the product of a phase factor $\exp\{-2i\beta[\ln 2pr\sin(\theta/2)+C]\}$, where C is Euler's constant, and a higher order correction factor composed of finite terms.

The perturbation calculation is carried out in Sec. II, while Sec. III deals with the summation of the partial-wave amplitudes. Section IV gives the correction factor to the relativistic Rutherford cross section. Natural units ($\hbar=c=1$) are used.

II. PERTURBATION CALCULATION

Applying the well-known rules[3] for writing the elements of the S matrix for a charged scalar particle, one finds for the first-order result $M^{(1)}$ and the second-order result $M^{(2)}=(1/2!)(2M_a{}^{(2)}+M_b{}^{(2)})$, $M_a{}^{(2)}$ having a diagram with two single corners and $M_b{}^{(2)}$ a double corner

$$M^{(1)}=-16\pi^2 iZe^2E/\mathbf{Q}^2, \tag{2}$$

$$M_a{}^{(2)}=16i(Ze^2)^2E^2I_a, \tag{3}$$

$$M_b{}^{(2)}=8i(Ze^2)^2I_b, \tag{4}$$

with the over-all squared momentum transfer

$$\mathbf{Q}^2=(\mathbf{p}'-\mathbf{p})^2=4p^2\sin^2(\theta/2), \tag{5}$$

I_a and I_b being integrals over three-momentum space.

The integral in Eq. (3) can be expressed as follows:

$$I_a=\int d^3q(\mathbf{q}^2-\mathbf{p}^2-i\epsilon)^{-1}[(\mathbf{q}-\mathbf{p})^2+\lambda^2]^{-1}[(\mathbf{q}-\mathbf{p}')^2+\lambda^2]^{-1}$$

$$=-\int_0^1 dx\frac{1}{2n}\frac{d}{dn}K(r,p,in), \tag{6}$$

with

$$\mathbf{r}=x\mathbf{p}+(1-x)\mathbf{p}',$$

$$n^2=x(1-x)\mathbf{Q}^2+\lambda^2,$$

$K(r,p,in)$ being given by Eq. (A4) of the Appendix. One gets after a short calculation

$$\lim_{\lambda\rightarrow 0} I_a=(\pi^2 i/p\mathbf{Q}^2)\ln(\mathbf{Q}^2/\lambda^2). \tag{7}$$

* Work supported in part by the National Science Foundation.

† This research forms part of a thesis to be submitted to the Graduate School of Northwestern University in partial fulfillment of the requirements for the Ph.D.

[1] R. H. Dalitz, Proc. Roy. Soc. (London) **206**, 509 (1951).

[2] C. Kacser, Nuovo cimento **13**, 303 (1959).

[3] S. S. Schweber, *Introduction to Quantum Field Theory* (Row, Peterson and Company, Evanston, Illinois, 1961).

The integral I_b in Eq. (4) is

$$I_b=\int d^3q[(\mathbf{q}-\mathbf{p})^2+\lambda^2]^{-1}[(\mathbf{q}-\mathbf{p}')^2+\lambda^2]^{-1}, \quad (8)$$

which can be evaluated in a straightforward manner to yield

$$\lim_{\lambda\to 0} I_b=\pi^3/|\mathbf{Q}|. \quad (9)$$

Thus, in the limit of small λ, we obtain for the second-order contributions

$$M_a^{(2)}=(Ze^2)^2(-16\pi^2E^2/p\mathbf{Q}^2)\ln(\mathbf{Q}^2/\lambda^2), \quad (10)$$

$$M_b^{(2)}=(Ze^2)^2 8i\pi^3/|\mathbf{Q}|. \quad (11)$$

Turning now to the third-order terms, we note that there exist twelve diagrams. The six which have three single corners each give identical contributions $M_a^{(3)}$ and the six which have a single and a double corner each give identical contributions $M_b^{(3)}$, where

$$M_a^{(3)}=(Ze^2)^3(-16iE^3/\pi^2)J_a, \quad (12)$$

$$M_b^{(3)}=(Ze^2)^3(-8iE/\pi^2)J_b, \quad (13)$$

with

$$J_a=\int d^3q_2(\mathbf{q}_2{}^2+\lambda^2)^{-1}L_a, \quad (14)$$

$$J_b=\int d^3q_2(\mathbf{q}_2{}^2+\lambda^2)^{-1}L_b, \quad (15)$$

L_a and L_b themselves being integrals over three-momentum space:

$$L_a=\int d^3q_1\{(\mathbf{q}_1{}^2+\lambda^2)(\mathbf{q}_1{}^2+2\mathbf{p}\cdot\mathbf{q}_1)[(\mathbf{Q}-\mathbf{q}_1-\mathbf{q}_2)^2+\lambda^2]\times[2\mathbf{p}\cdot(\mathbf{q}_1+\mathbf{q}_2)+(\mathbf{q}_1+\mathbf{q}_2)^2]\}^{-1}, \quad (16)$$

$$L_b=\int d^3q_1\{(\mathbf{q}_1{}^2-\mathbf{p}^2-i\epsilon)[(\mathbf{q}_1-\mathbf{p})^2+\lambda^2]\times[(\mathbf{p}'-\mathbf{q}_1-\mathbf{q}_2)^2+\lambda^2]\}^{-1}. \quad (17)$$

By combining the denominators in (16), L_a can be written

$$L_a=\int_0^1 dx\int_0^1 dy\frac{\partial}{\partial\Lambda_x{}^2}\frac{\partial}{\partial\Lambda_y{}^2}K(b,\Lambda_x,\Lambda_y), \quad (18)$$

with $K(b,\Lambda_x,\Lambda_y)$ again being given in Eq. (A4) of the Appendix and

$$\begin{aligned}\mathbf{b}&=y\mathbf{p}'-\mathbf{q}_2-x\mathbf{p},\\ \Lambda_x{}^2&=p^2(1-x)^2-\lambda^2x+i\epsilon(1-x),\\ \Lambda_y{}^2&=p^2(1-y)^2-\lambda^2y+i\epsilon(1-y).\end{aligned} \quad (19)$$

Since

$$L_a=\pi^2 i\int_0^1 dx\int_0^1 dy\frac{1}{\Lambda_y}\frac{\partial}{\partial\Lambda_x{}^2}\frac{1}{\mathbf{b}^2-(\Lambda_x+\Lambda_y)^2}, \quad (20)$$

we obtain from Eq. (14),

$$J_a=(\pi^2 i)\int_0^1 dx\int_0^1 dy\frac{1}{\Lambda_y}\frac{\partial}{\partial\Lambda_x{}^2}K(|y\mathbf{p}'-x\mathbf{p}|,i\lambda,\Lambda_x+\Lambda_y), \quad (21)$$

so that we finally get the symmetrical expression

$$J_a=(\pi^2 i)^2\int_0^1 dx\int_0^1 dy(\Lambda_x\Lambda_y)^{-1}\times[(y\mathbf{p}'-x\mathbf{p})^2-(\Lambda_x+\Lambda_y+i\lambda)^2]^{-1}. \quad (22)$$

As this integral has been evaluated by Kacser[2] we will merely quote his result,

$$J_a=-(2\pi^4/p^2\mathbf{Q}^2)\ln^2(|\mathbf{Q}|/\lambda). \quad (23)$$

Equation (17) may be evaluated using standard procedure as

$$L_b=\pi^2 i\int_0^1 dx\,\Lambda_x{}^{-1}[(\mathbf{q}_2-\mathbf{p}'+x\mathbf{p})^2-(\Lambda_x+i\lambda)^2]^{-1}, \quad (24)$$

where Λ_x is defined in Eq. (19). It follows immediately that

$$J_b=\pi^2 i\int_0^1 dx\,\Lambda_x{}^{-1}K(|\mathbf{p}'-x\mathbf{p}|,i\lambda,\Lambda_x+i\lambda). \quad (25)$$

In the Appendix it is shown that

$$J_b=\frac{\pi^4}{p|\mathbf{Q}|}\left[\frac{\pi i}{2}\ln(\mathbf{Q}^2/\lambda^2)+\pi i\ln\left(\frac{4}{1+\sin(\theta/2)}\right)+L_2\left(\sin\frac{\theta}{2}\right)-L_2\left(-\sin\frac{\theta}{2}\right)+\ln\sin\frac{\theta}{2}\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}\right], \quad (26)$$

with[4]

$$L_2(x)\equiv-\int_0^x\frac{\ln(1-t)}{t}dt.$$

We thus get

$$M_b^{(3)}=-16\pi^2 i(Ze^2)^3\frac{E}{\mathbf{Q}^2}\sin(\theta/2)\left[L_2\left(\sin\frac{\theta}{2}\right)-L_2\left(-\sin\frac{\theta}{2}\right)+\ln\sin\frac{\theta}{2}\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}+\frac{\pi i}{2}\ln(\mathbf{Q}^2/\lambda^2)+\pi i\ln\left(\frac{4}{1+\sin(\theta/2)}\right)\right]. \quad (27)$$

Summing the scattering amplitude up to the third order, assigning the correct statistical weights, we

[4] K. Mitchell, Phil Mag. 40, 351 (1949).

obtain

$$M=M^{(1)}+\frac{1}{2!}(2M_a^{(2)}+M_b^{(2)})+\frac{1}{3!}(6M_a^{(3)}+6M_b^{(3)})+\cdots$$

$$=-16\pi^2 i(Ze^2)\frac{E}{\mathbf{Q}^2}\left\{1-\beta\left[i\ln(\mathbf{Q}^2/\lambda^2)+\frac{\pi}{2}v^2\sin\frac{\theta}{2}\right]\right.$$
$$+\beta^2\left[-\tfrac{1}{2}\ln^2(\mathbf{Q}^2/\lambda^2)+\frac{\pi i}{2}v^2\sin\frac{\theta}{2}\ln(\mathbf{Q}^2/\lambda^2)\right.$$
$$+v^2\sin\frac{\theta}{2}\left(L_2\left(\sin\frac{\theta}{2}\right)-L_2\left(-\sin\frac{\theta}{2}\right)\right.$$
$$\left.+\ln\sin\frac{\theta}{2}\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}\right)$$
$$\left.\left.+\pi i v^2\sin\frac{\theta}{2}\ln\frac{4}{1+\sin(\theta/2)}\right]\right\}. \quad (28)$$

This can again be written as

$$M=M^{(1)}e^{-i\beta\ln(\mathbf{Q}^2/\lambda^2)}\left\{1-\beta\frac{\pi}{2}v^2\sin\frac{\theta}{2}+\beta^2v^2\sin\frac{\theta}{2}\right.$$
$$\times\left[\pi i\ln\frac{4}{1+\sin(\theta/2)}+\ln\sin\frac{\theta}{2}\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}\right.$$
$$\left.\left.+L_2\left(\sin\frac{\theta}{2}\right)-L_2\left(-\sin\frac{\theta}{2}\right)\right]\right\}. \quad (29)$$

Equation (29) shows explicitly that the terms containing $\ln\lambda$ can be represented as the expansion of a common phase factor.

III. COMPARISON WITH THE PARTIAL-WAVE SUM

It is known[5] that Coulomb scattering amplitude for the Klein-Gordon particle is given by the usual partial-wave sum:

$$f(\theta)=(2ip)^{-1}\sum_{l=0}^{\infty}(2l+1)(e^{2i\delta_l}-1)P_l(\cos\theta), \quad (30)$$

with the phase shifts

$$\delta_l=-\beta\ln 2pr+\eta_l+\frac{\pi}{2}(l+\tfrac{1}{2})-\frac{\pi}{2}[(l+\tfrac{1}{2})^2-\beta^2v^2]^{1/2}, \quad (31)$$

where

$$\eta_l=\arg\Gamma(l'+1+i\beta),\quad l'=-\tfrac{1}{2}+[(l+\tfrac{1}{2})^2-\beta^2v^2]^{1/2}, \quad (32)$$

[5] N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1950), 2nd ed.; L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company Inc., New York, 1955), 2nd ed; J. H. Hetherington, Ph.D. thesis (unpublished).

δ_l can be expanded in powers of β, yielding

$$f(\theta)=\frac{\beta}{p}\sum_{l=0}^{\infty}(2l+1)\{-\ln 2pr+\psi(l+1)\}P_l$$
$$+\frac{\beta^2}{p}\sum_{l=0}^{\infty}(2l+1)\left\{\frac{\pi}{2}\frac{v^2}{2l+1}+i[-\ln 2pr+\psi(l+1)]^2\right\}P_l$$
$$+\frac{\beta^3}{p}\sum_{l=0}^{\infty}(2l+1)P_l\left\{\frac{-1}{6}\psi''(l+1)-\frac{v^2}{2l+1}\psi'(l+1)\right.$$
$$+\frac{\pi i v^2}{2l+1}[-\ln 2pr+\psi(l+1)]$$
$$\left.-\tfrac{2}{3}[-\ln 2pr+\psi(l+1)]^3\right\}+O(\beta^4), \quad (33)$$

with

$$\psi(x)\equiv\frac{d}{dx}\ln\Gamma(x).$$

In order to evaluate the sums appearing in (33), we make use of the fact that the nonrelativistic Schrödinger amplitude for Coulomb scattering is known *both* in closed form and as a partial-wave sum

$$f_{\text{N.R.}}(\theta)=(2ip)^{-1}\sum_{l=0}^{\infty}(2l+1)\frac{\Gamma(l+1+i\beta)}{\Gamma(l+1-i\beta)}P_l(\cos\theta)$$
$$=-\frac{\beta}{2p}\left(\sin\frac{\theta}{2}\right)^{-2-2i\beta}\frac{\Gamma(1+i\beta)}{\Gamma(1-i\beta)}. \quad (34)$$

Expanding (34) in powers of β and making a term by term comparison we can, therefore, obtain the following relations:

$$\sum_{l=0}^{\infty}(2l+1)P_l\psi(l+1)=-\frac{1}{2\sin^2(\theta/2)}, \quad (35)$$

$$\sum_{l=0}^{\infty}(2l+1)P_l\psi(l+1)^2=\frac{1}{\sin^2(\theta/2)}\left[\ln\sin\frac{\theta}{2}+C\right], \quad (36)$$

$$\sum_{l=0}^{\infty}\frac{(2l+1)}{3}P_l\{-\tfrac{1}{2}\psi''(l+1)-2\psi(l+1)^3\}$$
$$=\frac{1}{\sin^2(\theta/2)}\left[\ln\sin\frac{\theta}{2}+C\right]^2,\ \text{etc.} \quad (37)$$

In addition, we need for the third-order contribution,

the results[6]

$$\sum_{l=0}^{\infty}\psi'(l+1)P_l=\frac{-1}{u}\left[\ln\frac{u}{2}\ln\frac{2+u}{2-u}-L_2\left(\frac{u}{2}\right)+L_2\left(-\frac{u}{2}\right)\right], \quad (38)$$

$$\sum_{l=0}^{\infty}\psi(l+1)P_l=\frac{-1}{u}\left[\ln\sin\frac{\theta}{2}+C+\ln\frac{4}{1+\sin(\theta/2)}\right],$$
$$\text{with } u=2\sin(\theta/2). \quad (39)$$

The foregoing, together with the well-known relations

$$\sum_{l=0}^{\infty}(2l+1)P_l=0,\quad \sum_{l=0}^{\infty}P_l=\frac{1}{2\sin(\theta/2)}\quad\text{for }\theta\neq 0, \quad (40)$$

make it possible to evaluate the sums appearing in (33), so that it becomes

$$f(\theta)=\frac{-\beta}{2p\sin^2(\theta/2)}e^{-2i\beta[C+\ln 2pr\sin(\theta/2)]}\times\left\{1-\frac{\beta\pi}{2}v^2\sin\frac{\theta}{2}+i\pi\beta^2v^2\sin\frac{\theta}{2}\ln\frac{4}{1+\sin(\theta/2)}+\beta^2v^2\sin\frac{\theta}{2}\left[\ln\sin\frac{\theta}{2}\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}+L_2\left(\sin\frac{\theta}{2}\right)-L_2\left(-\sin\frac{\theta}{2}\right)\right]\right\}, \quad (41)$$

which agrees with (29) providing we replace r by $(e^C\lambda)^{-1}$.

IV. CONCLUSION

We have verified to third order in $\beta=Ze^2/\hbar v$ that the infinite terms appearing in the Born expansion of the Coulomb scattering amplitude of a relativistic spinless particle are the terms of an expansion in β of a common phase factor.

The first-order relativistic Coulomb cross section has the finite correction factor

$$R=1-\beta\pi v^2\sin(\theta/2)+\tfrac{1}{4}\beta^2\pi^2v^4\sin^2(\theta/2)+2\beta^2v^2\sin(\theta/2)\left[\ln\sin\frac{\theta}{2}\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}+L_2\left(\sin\frac{\theta}{2}\right)-L_2\left(-\sin\frac{\theta}{2}\right)\right]. \quad (41)$$

---

[6] We are indebted to Dr. L. Maximon for the evaluation of the sums given in Eqs. (38) and (39) and we wish to thank him for his helpful interest.

APPENDIX

We evaluate first the integral over all three-momentum space,

$$K(r,s,t)=\int d^3q(\mathbf{q}^2-s^2-i\epsilon)^{-1}[(\mathbf{q}-\mathbf{r})^2-t^2]^{-1}, \quad (A1)$$

with s^2 real and $r=|\mathbf{r}|$. Combining denominators with the Feynman identity we get

$$K(r,s,t)=\int_0^1 dx\int d^3q(\mathbf{q}^2-D^2-i\epsilon)^{-2}, \quad (A2)$$

with

$$D^2=\mathbf{r}^2x^2-(\mathbf{r}^2+s^2-t^2)x+s^2. \quad (A3)$$

Using a spherical coordinate system and applying the method of residues one obtains easily

$$K(r,s,t)=\frac{\pi^2 i}{r}\ln\frac{s+t+r}{s+t-r}. \quad (A4)$$

Our second task is the evaluation of Eq. (25) of the text, which becomes

$$J_b=\frac{\pi^2 i}{p^2}\int_0^1\frac{dx}{\Lambda_x'}K(r',i\lambda',\Lambda_x'+i\lambda'), \quad (A5)$$

with

$$\Lambda_x'=\Lambda_x/p,\quad \lambda'=\lambda/p,\quad\text{and}\quad r'=|\mathbf{p}'-x\mathbf{p}|/p.$$

We put $\mathbf{p}\cdot\mathbf{p}'=p^2\cos\theta$, and in Λ_x let $\epsilon=0$ for fixed small λ, so that

$$\Lambda_x'=[(1-x)^2-\lambda'^2x]^{1/2},\quad r=(1-2x\cos\theta+x^2)^{1/2}. \quad (A6)$$

We next break up the integration region into

$$\text{(I) } 0\leqslant y\leqslant\lambda'-\tfrac{1}{2}\lambda'^2,\quad \text{(II) } \lambda'-\tfrac{1}{2}\lambda'^2\leqslant y\leqslant\lambda'^{1/2},\quad \text{(III) } \lambda'^{1/2}\leqslant y\leqslant 1, \quad (A7)$$

after replacing x by $1-y$. We let $u=2\sin(\theta/2)=|\mathbf{Q}|/p$.

By appropriate expansion of the integrand of (A5) with respect to λ', it follows immediately that for $\lambda'\ll 1$:

$$J_{b\mathrm{I}}=(\pi^4/p^2)(\pi^2/2u), \quad (A8)$$

$$J_{b\mathrm{II}}=(\pi^4/p^2)(\pi i/u)\ln(2/\lambda'^{1/2}), \quad (A9)$$

while

$$J_{b\mathrm{III}}=\frac{\pi^4}{p^2}\int_{\lambda'^{1/2}}^1\frac{dy}{y[y^2+u^2(1-y)]^{1/2}}\times\left\{\ln\frac{[y^2+u^2(1-y)]^{1/2}-y}{[y^2+u^2(1-y)]^{1/2}+y}+i\pi\right\}. \quad (A10)$$

The term containing $i\pi$ can be integrated to give

$$(\pi^4/P|\mathbf{Q}|)i\pi[\ln(1+\tfrac{1}{2}u)-\ln 2u-\tfrac{1}{2}\ln\lambda'], \quad (A11)$$

and the remaining part of $J_{b\mathrm{III}}$ can be simplified by the successive changes of variable:

$$y=u\sin w/\cos(w-\theta/2),$$
$$\sin w=\cos(\theta/2)\cos v, \quad (A12)$$
$$\tan(v/2)=z.$$

That is,

$$\lim_{\lambda'\to 0}\int_{\lambda'^{1/2}}^{1}\frac{dy}{y[y^2+u^2(1-y)]^{1/2}}\ln\frac{[y^2+u^2(1-y)]^{1/2}-y}{[y^2+u^2(1-y)]^{1/2}+y}$$

$$=\int_0^{\pi/2}\frac{\sin v dv}{u\cos v[1-\cos^2(\theta/2)\cos^2 v]^{1/2}}\ln\frac{1-\cos v}{1+\cos v}$$

$$=\int_0^1\frac{8zdz\ln z}{u(1-z^2)[(1+z^2)^2-\cos^2(\theta/2)(1-z^2)^2]^{1/2}}$$

$$=-\frac{1}{u}\left[\frac{\pi^2}{2}+L_2\left(-\frac{u}{2}\right)-L_2\left(\frac{u}{2}\right)+\ln\frac{u}{2}\ln\frac{2+u}{2-u}\right]. \quad \text{(A13)}$$

Summing the four expressions (A8), (A9), (A11), and (A13) we obtain finally

$$J_b=\frac{\pi^4}{p|\mathbf{Q}|}\left\{i\pi\ln(|\mathbf{Q}|/\lambda)+i\pi\ln\frac{4}{1+\sin(\theta/2)}\right.$$
$$+\ln\sin(\theta/2)\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}+L_2\left(\sin\frac{\theta}{2}\right)$$
$$\left.-L_2\left(-\sin\frac{\theta}{2}\right)\right\}. \quad \text{(A14)}$$

PHYSICAL REVIEW VOLUME 128, NUMBER 6 DECEMBER 15, 1962

Consequences of the Postulate of a Complete Commuting Set of Observables in Quantum Electrodynamics*

FREDERIK J. BELINFANTE
Department of Physics, Purdue University, Lafayette, Indiana
(Received May 21, 1962; revised manuscript received August 3, 1962)

It is shown that the radiation gauge (Coulomb gauge) of the potentials is the average of DeWitt's line-dependent gauge over all straight lines at constant time converging to the point where the potential is to be calculated. The radiation gauge is then used for demonstrating how the postulate of a complete commuting set of observables, contrary to Aharonov and Bohm's point of view, requires gauge-independent quantum electrodynamics rather than the use of the potentials of the Lorentz gauge.

1. INTRODUCTION. GAUGE DEPENDENCE

It is well known that the potentials A_μ ($\mathbf{A}$ with components A_n, and $\Phi=A^0=-A_0$) can be solved from

$$\mathbf{B}=\text{curl}\mathbf{A},\quad \mathbf{E}=-\nabla\Phi-\partial\mathbf{A}/c\partial t, \quad (1)$$

with an arbitrary choice

$$\phi=\text{div}\mathbf{A} \quad (2)$$

for the divergence of the vector potential, by

$$\mathbf{A}=\boldsymbol{a}+\nabla\Lambda,\quad \Phi=V-\partial\Lambda/c\partial t, \quad (3)$$

where

$$\boldsymbol{a}(\mathbf{x},t)=\int d^3z\,\text{curl}\mathbf{B}(\mathbf{z},t)/4\pi r, \quad \text{(4a)}$$

$$V(\mathbf{x},t)=\int d^3z\,\text{div}\mathbf{E}(\mathbf{z},t)/4\pi r=\int d^3z\,\rho(\mathbf{z},t)/r, \quad \text{(4b)}$$

$$\Lambda(\mathbf{x},t)=-\int d^3z\,\phi(\mathbf{z},t)/4\pi r, \quad \text{(4c)}$$

$$r=|\mathbf{z}-\mathbf{x}|.$$

The arbitrariness of the field $\phi=\nabla^2\Lambda$ leaves arbitrariness in $\mathbf{A}$ and Φ which is called the *gauge* dependence of the potentials. This arbitrariness can be removed only by making a *unique* choice for ϕ and, thence, for Λ, and then not allowing any different choice. The most natural way of achieving this, in a given Lorentz frame, is by postulating $\phi=\Lambda=0$. The potentials, then, are a^μ ($\boldsymbol{a}$ and V). This gauge is called the radiation gauge (as $\mathbf{B}=\text{curl}\boldsymbol{a}$ and $\boldsymbol{\mathcal{E}}=\mathbf{E}_\perp=-\partial\boldsymbol{a}/c\partial t$, with $\text{div}\mathbf{E}_\perp=0$ and with curl $\mathbf{E}_\perp=\text{curl}\mathbf{E}$, are said to represent the radiation field), or it is called the Coulomb gauge (as $\mathbf{E}_\parallel\equiv\mathbf{E}-\mathbf{E}_\perp=-\nabla V$ is the Coulomb field). A theory operating completely within this particular gauge is, therefore, called "gauge independent." In that case, V and $\boldsymbol{a}$ should not be regarded as independent variables, but as abbreviations for the integrals appearing in Eqs. (4a) and (4b), expressing them directly in terms of gauge-independent observables.

Let ψ be the "gauge-independent" wave function of some particles with charge e in the radiation gauge. Let Ψ be the gauge-dependent wave function for these particles in the arbitrary $\mathbf{A},\Phi$ gauge. Then, for consistency of the wave-mechanical description, one must have

$$\Psi=\exp[(ie/\hbar c)\Lambda]\psi. \quad (5)$$

2. DeWITT'S LINE-DEPENDENT GAUGE

In a recent article,[1] DeWitt has introduced a particular gauge depending on the choice of a set of space-

* Supported in part by the National Science Foundation.

[1] B. S. DeWitt, Phys. Rev. 125, 2189 (1962).