That is,

$$\lim_{\lambda'\to 0}\int_{\lambda'^{1/2}}^{1}\frac{dy}{y[y^2+u^2(1-y)]^{1/2}}\ln\frac{[y^2+u^2(1-y)]^{1/2}-y}{[y^2+u^2(1-y)]^{1/2}+y}$$
$$=\int_0^{\pi/2}\frac{\sin v dv}{u\cos v[1-\cos^2(\theta/2)\cos^2 v]^{1/2}}\ln\frac{1-\cos v}{1+\cos v}$$
$$=\int_0^1\frac{8z dz\ln z}{u(1-z^2)[(1+z^2)^2-\cos^2(\theta/2)(1-z^2)^2]^{1/2}}$$
$$=-\frac{1}{u}\left[\frac{\pi^2}{2}+L_2\left(-\frac{u}{2}\right)-L_2\left(\frac{u}{2}\right)+\ln\frac{u}{2}\ln\frac{2+u}{2-u}\right]. \quad \text{(A13)}$$

Summing the four expressions (A8), (A9), (A11), and (A13) we obtain finally

$$J_b=\frac{\pi^4}{p|\mathbf{Q}|}\left\{i\pi\ln(|\mathbf{Q}|/\lambda)+i\pi\ln\frac{4}{1+\sin(\theta/2)}\right.$$
$$+\ln\sin(\theta/2)\ln\frac{1-\sin(\theta/2)}{1+\sin(\theta/2)}+L_2\left(\sin\frac{\theta}{2}\right)$$
$$\left.-L_2\left(-\sin\frac{\theta}{2}\right)\right\}. \quad \text{(A14)}$$

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Consequences of the Postulate of a Complete Commuting Set of Observables in Quantum Electrodynamics*

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It is shown that the radiation gauge (Coulomb gauge) of the potentials is the average of DeWitt's line-dependent gauge over all straight lines at constant time converging to the point where the potential is to be calculated. The radiation gauge is then used for demonstrating how the postulate of a complete commuting set of observables, contrary to Aharonov and Bohm's point of view, requires gauge-independent quantum electrodynamics rather than the use of the potentials of the Lorentz gauge.

1. INTRODUCTION. GAUGE DEPENDENCE

IT is well known that the potentials A_μ ($\mathbf{A}$ with components A_n, and $\Phi=A^0=-A_0$) can be solved from

$$\mathbf{B}=\mathrm{curl}\mathbf{A},\quad \mathbf{E}=-\nabla\Phi-\partial\mathbf{A}/c\partial t, \quad (1)$$

with an arbitrary choice

$$\phi=\mathrm{div}\mathbf{A} \quad (2)$$

for the divergence of the vector potential, by

$$\mathbf{A}=\boldsymbol{a}+\nabla\Lambda,\quad \Phi=V-\partial\Lambda/c\partial t, \quad (3)$$

where

$$\boldsymbol{a}(\mathbf{x},t)=\int d^3z\,\mathrm{curl}\mathbf{B}(\mathbf{z},t)/4\pi r, \quad (4a)$$

$$V(\mathbf{x},t)=\int d^3z\,\mathrm{div}\mathbf{E}(\mathbf{z},t)/4\pi r=\int d^3z\,\rho(\mathbf{z},t)/r, \quad (4b)$$

$$\Lambda(\mathbf{x},t)=-\int d^3z\,\phi(\mathbf{z},t)/4\pi r, \quad (4c)$$

$$r=|\mathbf{z}-\mathbf{x}|.$$

The arbitrariness of the field $\phi=\nabla^2\Lambda$ leaves arbitrariness in $\mathbf{A}$ and Φ which is called the *gauge* dependence of the potentials. This arbitrariness can be removed only by making a *unique* choice for ϕ and, thence, for Λ, and then not allowing any different choice. The most natural way of achieving this, in a given Lorentz frame, is by postulating $\phi=\Lambda=0$. The potentials, then, are a^μ ($\boldsymbol{a}$ and V). This gauge is called the radiation gauge (as $\mathbf{B}=\mathrm{curl}\boldsymbol{a}$ and $\mathcal{E}=\mathbf{E}_\perp=-\partial\boldsymbol{a}/c\partial t$, with $\mathrm{div}\mathbf{E}_\perp=0$ and with curl $\mathbf{E}_\perp=\mathrm{curl}\mathbf{E}$, are said to represent the radiation field), or it is called the Coulomb gauge (as $\mathbf{E}_\parallel\equiv\mathbf{E}-\mathbf{E}_\perp=-\nabla V$ is the Coulomb field). A theory operating completely within this particular gauge is, therefore, called "gauge independent." In that case, V and $\boldsymbol{a}$ should not be regarded as independent variables, but as abbreviations for the integrals appearing in Eqs. (4a) and (4b), expressing them directly in terms of gauge-independent observables.

Let ψ be the "gauge-independent" wave function of some particles with charge e in the radiation gauge. Let Ψ be the gauge-dependent wave function for these particles in the arbitrary $\mathbf{A},\Phi$ gauge. Then, for consistency of the wave-mechanical description, one must have

$$\Psi=\exp[(ie/\hbar c)\Lambda]\psi. \quad (5)$$

2. DeWITT'S LINE-DEPENDENT GAUGE

In a recent article,[1] DeWitt has introduced a particular gauge depending on the choice of a set of space-

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[1] B. S. DeWitt, Phys. Rev. 125, 2189 (1962).

like lines leading toward all field points x^μ from infinity. Let $z^\mu(x,\xi)$ be a point on the line toward x, with ξ a parameter chosen in such a way that

$$z^\mu(x,0)=x^\mu \quad \text{and} \quad z^\mu(x,-\infty)=\text{spatial infinity}. \tag{6}$$

[Thence, $\partial z^\mu/\partial x^\lambda=\delta_\lambda{}^\mu$ at $\xi=0$.] Let a_μ and ψ temporarily denote the potentials and charged-particle wave function in some *arbitrary* gauge (not necessarily the radiation gauge), and let Λ be the gauge function transforming them according to (3) and (5) to some different gauge A_μ and Ψ. DeWitt then chooses

$$\Lambda(x)=-\int_{-\infty}^{0} a_\sigma(z)(\partial z^\sigma/\partial\xi)d\xi, \tag{7}$$

and shows that, by (3), this leads to

$$A_\mu(x)=\int_{-\infty}^{0} F_{\nu\sigma}(z)(\partial z^\nu/\partial\xi)(\partial z^\sigma/\partial x^\mu)d\xi \tag{8}$$

and[2]

$$\partial_\lambda A_\mu-\partial_\mu A_\lambda=\partial_\lambda a_\mu-\partial_\mu a_\lambda\equiv F_{\lambda\mu}. \tag{9}$$

In this gauge, the potentials depend not merely on the field strengths, but also on the set of lines chosen. A different set of lines leads to a different gauge. Therefore, we regard DeWitt's theory as not entirely gauge independent.

Take, for instance, lines $t=$const in a given Lorentz frame, so that $\partial z^0/\partial\xi=0$, and Eq. (7) reduces to

$$\Lambda(\mathbf{x})=-\int_{-\infty}^{\mathbf{x}} \boldsymbol{a}(\mathbf{z})\cdot d\mathbf{z}. \tag{10}$$

Let l be the line toward a given point $\mathbf{x}$, along which the integral in (10) is performed in one of DeWitt's gauges, and let the line l' toward $\mathbf{x}$ be used in a different choice of lines. Since the vector potentials $\boldsymbol{a}$ in points at the same time t all commute, we find from Eq. (5) that for the particle wave functions the two gauges are related by

$$\Psi'(\mathbf{x})=\exp[(ie/\hbar c)(\Lambda'-\Lambda)]\Psi(\mathbf{x})$$
$$=\exp\left[(ie/\hbar c)\left(\int_l-\int_{l'}\right)\boldsymbol{a}(\mathbf{z})\cdot d\mathbf{z}\right]\Psi(\mathbf{x}).$$

Here, $(\int_l-\int_{l'})$ is an integration from infinity one way toward $\mathbf{x}$ and back to infinity along a different path. We may close this path of integration to a loop L by a connection at spatial infinity, where all fields and potentials are assumed to vanish. Thence,

$$\Psi'(\mathbf{x})=\exp\left[(ie/\hbar c)\oint_L \boldsymbol{a}(\mathbf{z})\cdot d\mathbf{z}\right]\Psi(\mathbf{x})$$
$$=\exp\left[(ie/\hbar c)\int\int_S \mathbf{B}\cdot d\mathbf{S}\right]\Psi(\mathbf{x}), \tag{11}$$

where S is a surface having the lines l and l' as edges. Since in general the magnetic flux $\iint_S \mathbf{B}\cdot d\mathbf{S}$ will not vanish, Eq. (11) shows explicitly that the gauge of $\Psi(\mathbf{x})$ depends on the choice of the line l toward $\mathbf{x}$, and that it is *incorrect* to call $\Psi(\mathbf{x})$ a "gauge-invariant" wave function.[3] As long as there is no *unique* way of fixing the choice of the lines once and for all (as long as we can choose *different* lines l), there remains the dependence of the gauge on the arbitrariness of DeWitt's set of lines.[4]

3. RELATION OF THE RADIATION GAUGE TO DeWITT'S LINE-DEPENDENT GAUGES

We shall start here from DeWitt's line-dependent gauge, and then in a given Lorentz frame make the gauge unique by a particular choice of the lines. First of all, we confine these lines to straight lines at $t=$const, converging toward the field points $\mathbf{x}$ in arbitrary directions given by unit vectors $\boldsymbol{\epsilon}$. Thus,

$$z^\nu=x^\nu+\epsilon^\nu\xi, \quad \epsilon^0=0, \quad \epsilon_n\epsilon^n\equiv\boldsymbol{\epsilon}^2=1. \tag{12}$$

Thence,

$$\partial z^\nu/\partial\xi=\epsilon^\nu, \quad \partial z^\sigma/\partial x^\mu=\delta_\mu{}^\sigma, \tag{13}$$

and Eq. (8) reduces to

$$A_\mu(x)=\int_{-\infty}^{0} F_{n\mu}(z)\epsilon^n d\xi. \tag{14}$$

Here,

$$-\xi=r\equiv|\mathbf{z}-\mathbf{x}| \tag{15}$$

is the radial distance of the point $\mathbf{z}$ from the field point $\mathbf{x}$.

The potential (14) still depends on the direction of the unit vector $\boldsymbol{\epsilon}$, which points inward, *toward* the point $\mathbf{x}$. To remove this dependence, let us take for the potentials the gauge obtained by *averaging* Eq. (14) over all possible directions of $\boldsymbol{\epsilon}$. The line integral (14) then becomes a volume integral over all space. In this volume integral, the lines within an infinitesimal solid angle $d\Omega$ will fill up a cone with its vertex at $\mathbf{x}$ and with volume elements

$$d^3z=r^2dr d\Omega. \tag{16}$$

The transition from the line integral in (14) to the volume integral is most easily performed by first inserting in (14) the identity

$$|d\xi|=|dr|=d^3z/r^2d\Omega, \tag{17}$$

and then averaging the resulting expression for A_μ

[2] Starting from Eq. (8), one can easily prove Eq. (9) *directly*, without use of Eq. (7) and of $A_\mu=a_\mu+\partial_\mu\Lambda$, by use of Eq. (6), of $F_{\nu\sigma}=-F_{\sigma\nu}$, of an integration by parts, and of $\partial_\rho F_{\nu\sigma}+\partial_\sigma F_{\rho\nu}+\partial_\nu F_{\sigma\rho}=0$.

[3] Reference 1, above its Eq. (6).

[4] If throughout DeWitt's theory $A_\mu(x)$ is regarded as simply an abbreviation for the potential-independent integral in Eq. (8), one may, of course, call DeWitt's theory "path dependent" instead of "gauge dependent."

over all possible directions of $\boldsymbol{\varepsilon}$ by

$$\bar{A}_\mu(x)=\int\int A_\mu(x)d\Omega/4\pi=\int\int\int d^3z\,F_{n\mu}(z)\epsilon^n/4\pi r^2$$
$$=\int\int\int d^3z\,F_{n\mu}(z)[x^n-z^n]/4\pi r^3 \tag{18}$$
$$=\int\int\int d^3z\,F_{n\mu}(z)(\partial/\partial z^n)(1/4\pi r).$$

We integrate this by parts and find

$$\bar{A}^\mu(x)=\int\int\int d^3z[\nabla_n F^{\mu n}(z)]/4\pi r. \tag{19}$$

If we write this out in components, using

$$F^{01}=-F_{01}=E_x,\quad F^{11}=0,\quad F^{12}=F_{12}=B_z,$$

etc., we find

$$\bar{A}^0(\mathbf{x},t)=\int\int\int d^3z[\mathrm{div}\mathbf{E}(\mathbf{z},t)]/4\pi r$$
$$\equiv a^0(\mathbf{x},t)\equiv V(\mathbf{x},t), \tag{20a}$$

and the spatial components of (19) form the vector

$$\bar{\mathbf{A}}(\mathbf{x},t)=\int\int\int d^3z[\mathrm{curl}\mathbf{B}(\mathbf{z},t)]/4\pi r\equiv\boldsymbol{a}(\mathbf{x},t). \tag{20b}$$

Thus we find back the radiation gauge of the potentials, if by this averaging procedure we remove the line dependence from DeWitt's gauge.

This question now poses itself: What happens to the wave functions of charged particles under this averaging procedure? It is easier to first answer the opposite question: How are the gauges of DeWitt's wave functions Ψ related to the "gauge-independent" radiation gauge ψ of these wave functions? The answer is given by Eq. (5) with Λ from Eq. (7), this time with ψ and a_σ taken in the gauge-independent gauge. Conversely, then, ψ is obtained from any of DeWitt's Ψ by

$$\psi(x)=\exp\left[(ie/\hbar c)\int_{-\infty}^{0}a_\sigma(z)(\partial z^\sigma/\partial\xi)d\xi\right]\Psi(x), \tag{21}$$

with the a^μ given gauge independently by Eqs. (4a) and (4b) or by Eq. (19). Since the radiation gauge is unique, the $\psi(x)$ that goes into the right-hand member of Eq. (5) and that, therefore, comes out of the left-hand member of Eq. (21) is unambiguous except for an arbitrary trivial constant phase factor $e^{i\alpha}$. This means that in Eq. (21) the line dependence of the integral is canceled by the line dependence of the gauge of $\Psi(x)$.

The gauge ψ for the charged-particle wave functions is then used together with the averaged gauge $\bar{A}_\mu=a_\mu$ for the potentials. This ψ follows from Ψ not by averaging, but by Eq. (21).

4. GAUGE-INDEPENDENT QUANTUM ELECTRODYNAMICS

The radiation gauge allows us to develop a quantum-electrodynamical theory entirely in terms of gauge-independent quantities.[5,6] In this theory, the $\mathbf{B}$ field and the transverse (=solenoidal) part $\boldsymbol{\mathcal{E}}$ of the $\mathbf{E}$ field are quantized in the conventional way. The longitudinal (Coulomb) part of the $\mathbf{E}$ field is given by the identity

$$\mathbf{E}_{||}(\mathbf{x},t)=-\boldsymbol{\nabla}V(\mathbf{x},t)=-\boldsymbol{\nabla}\int\int\int d^3z\,\rho(\mathbf{z},t)/r, \tag{22}$$

and is quantized only indirectly, by the quantization of the charged-matter fields ψ in terms of which ρ can be expressed. Thus, for instance, $\mathbf{E}_{||}$ as well as $\mathbf{E}=\mathbf{E}_{||}+\boldsymbol{\mathcal{E}}$ will satisfy the following commutation relation with the second-quantized electron wave function ψ:

$$[\mathbf{E}(\mathbf{x},t);\psi(\mathbf{x}',t)]=-\boldsymbol{\nabla}\int\int\int d^3z[\rho(\mathbf{z},t);\psi(\mathbf{x}',t)]/r$$
$$=e\int\int\int d^3z\,\delta(\mathbf{x}'-\mathbf{z})\psi(\mathbf{z},t)\boldsymbol{\nabla}(1/r) \tag{23}$$
$$=e\psi(\mathbf{x}',t)(\partial/\partial\mathbf{x})(1/|\mathbf{x}-\mathbf{x}'|)\neq 0.$$

This type of quantization avoids disagreement with Maxwell's equation $\mathrm{div}\mathbf{E}=4\pi\rho$.

In gauge-independent quantum electrodynamics, there are several ways of choosing a complete commuting set of observables α, so that an arbitrary state $\boldsymbol{\Psi}$ may always be expressed as a superposition

$$\boldsymbol{\Psi}=\textstyle\sum_\alpha C_\alpha\boldsymbol{\Psi}_\alpha. \tag{24}$$

The importance of the existence of such a set of observables was stressed by Aharonov and Bohm[7] in their criticism of DeWitt's work.

One can, for instance, use the matter fields ψ and the magnetic field $\mathbf{B}$ at some initial time t as a complete commuting set of variables. One may also choose the Fourier components of these fields. Since $\mathrm{div}\mathbf{B}=0$, there will be only two independent polarization directions for the Fourier components of $\mathbf{B}$ with any one given "wave vector" $\mathbf{k}$.

Instead of the $\mathbf{B}$ field or its Fourier components, some people may prefer to use the $\boldsymbol{a}$ field. As $\mathrm{div}\boldsymbol{a}=0$, also the $\boldsymbol{a}$ field contains transverse waves only.

Finally, one may prefer to use the occupation number representation, in which occupation numbers are constructed out of the Fourier components of these fields and their canonical conjugates. The fact that the re-

[5] F. J. Belinfante and J. S. Lomont, Phys. Rev. **84**, 541 (1951).
[6] C. L. Hammer and R. H. Good Jr., Ann. Phys. (New York) **12**, 463 (1961).
[7] Y. Aharonov and D. Bohm, Phys. Rev. **125**, 2192 (1962).

lation between fields and their Fourier components or occupation numbers is nonlocal does not make the Fourier components or the occupation numbers less useful. There are nowadays even people who claim that the latter would be "more fundamental" (whatever that might mean) than the fields themselves.[8]

Aharonov and Bohm[7] and DeWitt[1] have raised the academic question whether quantum electrodynamics, at least in its gauge-independent version, would be a local theory or a nonlocal theory. DeWitt points out that the answer depends on what one means by the word "local." Is, for instance, the relation between $\boldsymbol{a}$ and $\mathbf{B}$ local or not? This relation may be written either in the "local" form $\mathbf{B}=\mathrm{curl}\boldsymbol{a}$, or in the nonlocal form of Eq. (4a). What does it really matter how we label such a relation?

Suppose we would want to put the gauge-independent theory of electrons interacting with electromagnetic fields in a "canonical" form—not because this would be a necessity, but because we like it. As we have shown in the past,[5] this can be done by choosing the fields $\boldsymbol{a}$ and ψ (or their Fourier components) as the basic field coordinates $q(x)$, and the fields $(-\mathfrak{E}/4\pi c)$ and $(i\hbar\psi^\dagger)$ as their canonical conjugates $p(x)$. The Coulomb field $\mathbf{E}_{\parallel}$ in that case must be treated as merely a derived variable defined by Eq. (22) with $\rho=:e\psi^\dagger\psi:$. This procedure of quantization *was proved by us to be Lorentz covariant*, even though it is not manifestly so.[5] In the Heisenberg picture, $\mathbf{B}$ is found to form a tensor together with $\mathbf{E}$, while in the interaction picture it forms a tensor together with $\mathfrak{E}$. For an explanation of how that is possible, the reader is referred to the original paper.[5] For those who do not like the notation used in the original paper, there is a different notation introduced by Hammer and Good.[6]

While in a canonical formalism we use $\boldsymbol{a}$ as a canonical variable $q(x)$, this does *not* mean that $\boldsymbol{a}$ must always be regarded as an independent variable. In the first place, one might altogether avoid introducing canonical variables. As stated well by DeWitt,[1] one could introduce the quantization by simply writing down the (noncanonical) conventional commutation relations for $\mathfrak{E}$ and $\mathbf{B}$, which have been known from the early days of the quantum theory of radiation.[9] If we later would introduce $\boldsymbol{a}$ at all, as an abbreviation for the integral appearing in Eq. (4a), then the commutation relations of $\boldsymbol{a}$ and $\mathfrak{E}$ can be *derived* from those of $\mathbf{B}$ and $\mathfrak{E}$. In this way, $\boldsymbol{a}$ loses its "fundamental" character, and its canonical commutation relations $[qp-pq=i\hbar\delta]$ with $(-\mathfrak{E}/4\pi c)$ are treated as a happenstance.

As to V, from its quantization through the ψ fields it is much clearer that this is merely a nonlocally derived variable.

Also in the variational principle from which the field equations of gauge-independent quantum electrodynamics can be derived, the potentials $\boldsymbol{a}$ and V are not regarded as independent variables. These field equations follow from the variational principle[10]

$$\delta\int \mathfrak{L}dt=0 \tag{25}$$

with

$$\mathfrak{L}=-\int\bar{\psi}(mc^2+\hbar c\gamma^\mu\partial_\mu)\psi+\int\boldsymbol{a}\cdot(\partial\mathfrak{E}/4\pi c\partial t)$$
$$-\int(\mathfrak{E}^2+\mathbf{B}^2)/8\pi+\int\boldsymbol{a}\cdot\mathbf{j}-\tfrac{1}{2}\int\rho V, \tag{26}$$

where $\int\equiv\iiint d^3x$, and where ρ, $\mathbf{j}$, $\boldsymbol{a}$, and V are not varied independently, but depend on the independent variables ψ, $\bar{\psi}$, $\mathfrak{E}$, and $\mathbf{B}$ (with $\mathrm{div}\mathfrak{E}\equiv\mathrm{div}\mathbf{B}\equiv 0$) by means of

$$\begin{aligned}\rho&\equiv e:\psi^\dagger\psi:\equiv ie:\bar{\psi}\gamma^0\psi:,\\ \mathbf{j}&\equiv e:\psi^\dagger\boldsymbol{\alpha}\psi:\equiv ie:\bar{\psi}\boldsymbol{\gamma}\psi:,\end{aligned} \tag{27}$$

while $\boldsymbol{a}$ and V are given by the integrals in Eqs. (4a) and (4b). The last two terms in Eq. (26), therefore, are really *double* integrals over three-dimensional space. In this sense, the Lagrangian of gauge-independent quantum electrodynamics is really *nonlocal*.

The Coulomb term

$$-\tfrac{1}{2}\int\rho V\equiv-\tfrac{1}{2}\int\int'\rho(\mathbf{x})\rho(\mathbf{x}')/|\mathbf{x}-\mathbf{x}'| \tag{28}$$

would be nonlocal in form even if one would express it in terms of canonical variables (ψ and $\psi^\dagger$).

On the other hand, when expressed in terms of the potentials, the *field equations*

$$\mathbf{E}=-\boldsymbol{\nabla}V-\partial\boldsymbol{a}/c\partial t,\quad \mathbf{B}=\mathrm{curl}\boldsymbol{a}, \tag{29}$$

$$i\hbar\partial\psi/\partial t=(mc^2\beta-i\hbar c\boldsymbol{\alpha}\cdot\boldsymbol{\nabla}-e\boldsymbol{a}\cdot\boldsymbol{\alpha})\psi+\tfrac{1}{2}e(V\psi+\psi V), \tag{30}$$

seem to be local equations. The nonlocality, therefore, mainly enters through the quantization of the fields. [See, for instance, Eq. (23).]

5. CONSEQUENCES OF THE POSTULATE OF A COMPLETE COMMUTING SET OF OBSERVABLES IN QUANTUM ELECTRODYNAMICS

After stressing the importance of the existence of a complete commuting set of observables, Aharonov and Bohm[7] complain that DeWitt's theory[1] is a *nonlocal* theory, and that, to their knowledge, "no example of a consistent over-all formulation of a nonlocal theory in terms of *some* complete commuting set of observables has been given." They conclude that "the actual situation is then that the only electrodynamical theories that have thus far received any kind of over-all formulation in terms of a complete set of operators are

[8] Compare H. P. Stapp, Phys. Rev. **125**, 2139 (1962).

[9] In doing so, we conveniently forget that, historically, the commutation relations for $\mathfrak{E}$ and $\mathbf{B}$ were originally discovered by applying *canonical quantization* to certain "harmonic oscillators" describing the fields.

[10] F. J. Belinfante, Phys. Rev. **84**, 546 (1951).

these which are *essentially local* and which must involve potentials when their local character is clearly exhibited."

Again, one may quarrel about the meaning of the word "essentially" in front of "local" in the last sentence, but, if with Aharonov and Bohm one regards the direct and instantaneous Coulomb interaction featured by (25) through (28) as an "essentially" *non*local feature,[11] then gauge-independent quantum electrodynamics is a case in point to illustrate how incorrect Aharonov and Bohm's claims here are. We have seen in the preceding section that in gauge-independent quantum electrodynamics there exist *several* complete commuting sets of variables, of which each may serve to form by its eigenfunctions a basis for a Hilbert space (*with positive definite metric*), even though this theory is quite as nonlocal as DeWitt's from of quantum electrodynamics. Moreover, if we take the occupation numbers as the variables, many people will regard them as *observables*. (The particle wave functions ψ themselves can hardly be called observables, in particular not for fermions.)

Not only, then, *do* examples exist of nonlocal theories with a complete commuting set of observables, but the date 1951 of reference 5 shows that also the second statement of Aharonov and Bohm about the "actual situation" is incorrect.

On the other hand, as already stated by DeWitt[1] in his reply to Aharonov and Bohm, *conventional* manifestly Lorentz-covariant quantum electrodynamics, based on Fermi's modification of Maxwell's theory, may have a complete set of commuting *operators* as its basis, but those operators definitely *do not all correspond to observables*. Even in the occupation number representation, the numbers of scalar and of longitudinal photons appearing in that theory certainly are not observable. Worse yet, the general state $\mathbf{\Psi}$ expressed as a superposition $\sum_\alpha C_\alpha \mathbf{\Psi}_\alpha$ of eigenfunctions of the commuting quantities α, cannot be an arbitrary superposition, but is subjected to auxiliary conditions, so that the space determined by the set α in this case contains nonphysical parts. This leads to well-known difficulties, of which one is the need for an indefinite metric, so that the space with basis $\mathbf{\Psi}_\alpha$ is then not even a Hilbert space.[12]

We, therefore, conclude that gauge-independent quantum electrodynamics, though having nonlocal characteristics, is superior to conventional manifestly covariant local quantum electrodynamics and its potential four-vector in the Lorentz gauge, in as far as one looks for a theory based on a complete commuting set of *observables* determining a *Hilbert space*.

This, of course, does not mean that the conventional local theory would not possess some different properties for which its use for certain particular purposes might be preferable over the gauge-independent theory. For instance, nobody so far has cared to work out a complete theory of renormalization and of regularization in the radiation gauge.[13] Also in questions of causality, the use of the retarded potentials in the Lorentz gauge would seem to have definite advantages over the use of an instantaneously acting Coulomb potential and of a retarded *transverse* vector potential, which follows from the conventional retarded vector potential of the Lorentz gauge by a *nonlocal* operation.

As to DeWitt's *line-dependent* quantum electrodynamics, it has its formal beauty, but the arbitrariness introduced by the line dependence of his gauge seems to make this type of quantum electrodynamics more complicated than the all-out gauge-independent variety,[14] although one gains in manifest covariance.

Even if one rejects DeWitt's field of paths to field points, one must admit the validity of DeWitt's objection against Aharonov and Bohm's assertion about a supposed *necessity* for using potentials or for using a

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[11] However, in the field equations, only the ones for ψ and $\psi^\dagger$ become nonlocal, and ψ and $\psi^\dagger$ are not observables. Equations for *observables*, such as the kinetic momentum, involve only the field strengths. The analog to the local Lorentz force formula of classical mechanics is not a field equation, but is the gauge-independent c-number equation

$$(d/dt)\int\psi^\dagger(-i\hbar\nabla - e\boldsymbol{a}/c)\psi d^3x = e\int\psi^\dagger(\mathbf{E}+\boldsymbol{\alpha}\times\mathbf{B})\psi d^3x,$$

with $\mathbf{E}=\boldsymbol{\mathcal{E}}-\nabla V$ and with $\boldsymbol{a}$ and V given by Eqs. (4a–b).

[12] In gravitational theory, where a similar choice exists between a manifestly covariant theory with auxiliary conditions and a theory operating merely in a Hilbert space of allowable physical states, Anderson has found it impossible in the manifestly covariant theory to express consistently the auxiliary conditions as compatible conditions imposed simultaneously on a state vector, if the ordering of factors in the q-number theory is taken seriously.

[13] It is true that the very first correct relativistic calculation of the Lamb shift, published by J. B. French and V. F. Weisskopf in Phys. Rev. **75**, 1240 (1949), apparently uses the radiation gauge. However, they smuggled in the Lorentz gauge through the back door, by using a Fourier expansion of their first-order direct Coulomb interaction, and then combining the resulting integral over an apparent "wave vector" $\mathbf{k}$ with the integral over the wave vectors $\mathbf{k}$ of intermediate photons in the second-order radiative interaction, *before* performing these integrations. This artifice gives the formulas a form that would come natural only if the Coulomb interaction, too, were considered a second-order effect resulting from intermediate photons relativistically forming some kind of a unity with the transverse photons. *Only with the help of this trick* were they able to recognize which (by itself diverging) part of these integrals should be thrown out as representing merely a mass renormalization. The difficulty here is caused by the divergency of the separate integrals, which is a consequence, not of the use of this or that gauge of the potentials, but of the use of a Schrödinger equation for the state vector, which makes it impossible to avoid the occurrence of nonphysical self-interactions besides the physical ones that cause the Lamb shift. For the alternative to this, viz., the new possibilities in quantum field theory if one postulates merely *integrocausality*, see the attempts in F. J. Belinfante, Phys. Rev. **85**, 468 (1952), with the warning in Phys. Rev. **91**, 1285 (1953) pointing out the necessity of reformulating those ideas.

[14] S. Mandelstam, at the International Conference on Relativistic Theories of Gravitation, Jabłonna, Poland, July, 1962 (unpublished), has suggested the possibility of a line-dependent definition of gravitational field variables too. Here, with Mandelstam's interpretation of the variation of the paths relative to the path itself rather than relative to background coordinates, the additional complication arises that in curved space there is then no simple way of verifying whether a given variation of the path leading to a field point is going to change the location of this point or not.

local notation. In DeWitt's argument, one then merely has to replace his line-dependent gauge by our "gauge-independent" radiation gauge, which makes the auxiliary quantities a_μ depend merely on the field strengths $F_{\nu\sigma}$ and the Lorentz frame, without introducing any arbitrary elements that would make the potentials become quantities that depend on some other variables than the field strengths alone.

6. CONCLUDING REMARK

I hope that the above has brought out the following. In electrodynamics, one has a free choice between working with potentials or working with field strengths. More precisely, at least three choices are possible: (1) a local, manifestly covariant theory based on a set of operators α which contain physically redundant variables and of which the eigenfunctions Ψ_α do not form an ordinary Hilbert space; (2) a quasi-local theory, which in its field equations hides nonlocal features by the use of abbreviations like a_μ in the radiation gauge or A_μ for DeWitt's line-dependent gauge, but which cannot hide its nonlocality in its quantization; and (3) an openly nonlocal theory, which admits that the potentials are merely abbreviations for certain integrals of the field strengths. Here "nonlocal" refers to field equations for the field variables used, though these variables need not be observables. Choices (2) and (3) avoid nonphysical annexes to Hilbert space and allow the use of actual *observables* as a basic complete set of commuting variables α. The choice between the possibilities (1), (2), and (3), however, is largely a question of *convenience* for whatever use of the theory one has in mind. The idea must be dispelled that the same choice should necessarily be made under all circumstances and for all purposes. The mathematical advantages of the conventional local theory may sometimes outweigh the physical and metaphysical advantages of formulating the theory in the apparently more realistic Hilbert space of gauge-independent quantum electrodynamics.

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Volume Dependence of the Energy of an Enclosed Quantum Mechanical System

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A sum rule is developed that allows one to write a closed form expression for d^2E_n/dV^2; where $E_n(V)$ is the nth eigenvalue and V the volume of an enclosed many-body system with arbitrary interaction potential. This results in a simple differential equation for $E_n(V)$. Unfortunately, the solution, $E_n(V)$, does not appear to be physically meaningful. The reason for this is the appearance of unacceptable functions in the formal development of the sum rule. Because these functions do not belong to the linear manifold with respect to which the Hamiltonian is Hermitian, considerable care must be exercised in applying reduction formulas. Several examples are considered in detail.

I. INTRODUCTION

IN this paper we are concerned with Hermitian operators. Our attention, however, is directed, in particular, to those functions that do not satisfy Eq. (1) below. To begin, consider the following definition[1]: Let Ψ and Φ be *any* two "acceptable" functions defined in a certain region of configuration space τ; an Hermitian operator $\mathbf{O}$, then, has the property that

$$\int_\tau \Psi^*\mathbf{O}\Phi d\tau = \int_\tau \Phi\mathbf{O}^*\Psi^* d\tau. \tag{1}$$

The most crucial part of this definition of an Hermitian operator involves the word "acceptable." The latter usually implies that Ψ and Φ vanish sufficiently strongly on the surface that encloses the region τ of configuration space. The functions Ψ and Φ are also assumed to be quadratically integrable. A very thorough definition of "acceptable" function is given, for example, by Kemble.[2] Thus, his class D functions must, among other things, vanish at infinity faster than any negative power of the coordinates. The latter is also true of the first two derivatives and the functions and their derivatives must also be quadratically integrable.

Of course, less stringent requirements on the wave functions still preserve a Hermitian character. For example, if $\mathbf{O}$ has the form

$$\mathbf{O} \equiv \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x), \tag{2}$$

and a linear manifold of functions $\Psi_1, \Psi_2, \cdots$ exists such

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[1] H. Margenau and G. M. Murphy, *The Mathematics of Physics of Chemistry* (D. Van Nostrand Company, Inc., Princeton, New Jersey, 1956), 2nd ed., p. 269.

[2] E. C. Kemble, *Quantum Mechanics*, (Dover Publications, New York, 1958), p. 197.