# Asymptotic Symmetries in Gravitational Theory*

R. SACHS†
*U. S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey*
(Received July 23, 1962)

It is pointed out that the definition of the inhomogeneous Lorentz group as a symmetry group breaks down in the presence of gravitational fields even when the dynamical effects of gravitational forces are completely negligible. An attempt is made to rederive the Lorentz group as an "asymptotic symmetry group" which leaves invariant the form of the boundary conditions appropriate for asymptotically flat gravitational fields. By analyzing recent work of Bondi and others on gravitational radiation it is shown that, with apparently reasonable boundary conditions, one obtains not the Lorentz group but a larger group. The name "generalized Bondi-Metzner group" ("GBM group") is suggested for this larger group.

It is shown that the GBM group contains an Abelian normal subgroup whose factor group is isomorphic to the homogeneous orthochronous Lorentz group; that the GBM group contains precisely one Abelian four-dimensional normal subgroup, which can be identified with the group of rigid translations; that the GBM group contains an infinite number of different subgroups isomorphic to the inhomogeneous orthochronous Lorentz group; that the infinitesimal GBM group algebra permits at least one nontrivial representation, which is directly analogous to the rest-mass-zero and spin-zero representation of the Lorentz group; that in any representation of the infinitesimal GBM group algebra there is a "rest mass" operator which commutes with all the other operations; and that no similar "spin" operator appears to exist. It is argued that the GBM group is so similar to the inhomogeneous Lorentz group that the former may be compatible as a symmetry group with present microphysics.

Two applications are given: Certain quantum commutation relations covariant under GBM transformations are presented; and a denumerably infinite set of integral invariants, for classical asymptotically flat gravitational fields, are derived. The four simplest integral invariants constitute the total energy momentum radiated to infinity by gravitational waves.

## I. INTRODUCTION

If one neglects gravitational effects, the Lorentz group can be defined in one of two ways: (1) Lorentz transformations leave invariant the basic differential equations of microphysics; (2) Lorentz transformations are "symmetry" transformations that preserve the numerical value of the metric tensor. If one now takes into account the often extremely small gravitational effects via general relativity, one is faced with a peculiar situation. Every conceivable coordinate transformation has the first property mentioned. In the generic case, no coordinate transformation has the second property.[1] Thus only the homogeneous Lorentz transformations seem to remain conceptually well defined, as those transformations relevant in discussing properties at a fixed space-time point; the inhomogeneous Lorentz group seems to disappear into thin air.

However, the inhomogeneous Lorentz group plays a fundamental role in microphysics; it limits the possible types of elementary particles and its existence leads to conservation laws. Therefore, one cannot accept the notion that this group is eliminated by the presence of dynamically negligible gravitational fields. There must be some reasonable sense in which the Lorentz transformations are "approximate symmetry" transformations; and in microphysics the approximation must be a very accurate one. Now probably the most reasonable way to introduce the notion of an *approximate* symmetry in curved space-time is to consider *asymptotic* symmetries in a space-time whose metric is asymptotically Minkowskian.[1,2] One is therefore led to the questions: What, specifically, does it mean to say that a metric is asymptotically flat? What are the coordinate transformations that preserve the form of the appropriate boundary conditions?

Much work has been done on the question: What are physically sensible boundary conditions to place on the gravitational field?[1,2] The mathematical problem is one in global Riemannian geometry and, therefore, quite difficult. No really definitive results have been obtained. However, two very extensive and detailed treatments have been given recently, one by Bondi and his co-workers[3] (for axially symmetric fields) and one by Arnowitt, Deser, and Misner.[4] The approach due to Bondi was subsequently generalized and related to the theory of the Petrov-Pirani classification.[5-8] In

* Research supported jointly by the U. S. Army (U. S. Army Signal Research and Development Laboratory) and U. S. Air Force (ARL).

† Present address: Department of Physics, Stevens Institute, Hoboken, N. J.

[1] A. Trautman's "Lectures in General Relativity," King's College, London University, 1958, mimeographed notes (unpublished) contain a summary of Trautman's own work, a lucid and balanced survey of other work done prior to 1958, and a very excellent bibliography. Although these lectures are obtainable only by writing to the Department of Mathematics at King's their influence on the field has been so decisive that they must serve as the main reference for the ideas presented here. Another important paper is Trautman's article that is scheduled to appear in 1962 in a survey volume by L. Witten.

[2] Compare Trautman, reference 1; V. Fock, *Theory of Space, Time and Gravitation* (Pergamon Press, New York, 1959); R. Arnowitt, R., S. Deser, and C. W. Misner, Phys. Rev. **122**, 997 (1961) (which contains references to the previous papers by these authors). These are the references in which this idea is presented most clearly.

[3] H. Bondi, M. G. J. Van der Burg, and A. W. K. Metzner, Proc. Roy. Soc. (London) (to be published).

[4] R. Arnowitt, S. Deser, and C. W. Misner, reference 2.

[5] R. Sachs, Proc. Roy. Soc. (London) (to be published).

[6] E. T. Newman and R. Penrose (to be published).

[7] T. Unti and E. T. Newman (to be published); the fact that these authors use a preferred parameter distance rather than a luminosity distance is not important to our discussion.

[8] For the theory of the Petrov classification see the classic paper by F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957). Two reasonably up to date papers with fairly extensive bibliographies are P. Jordan, J. Ehlers, and R. Sachs, Akad. Wiss. Mainz (1) 1961,

this paper the Bondi approach, in which light-like ("null") hypersurfaces play a leading role, will be followed. However, the basic assumptions in the Arnowitt-Deser-Misner approach seem quite similar. It seems reasonable to hope that the two approaches are essentially equivalent.

One virtue of the approach due to Bondi is that no *a priori* assumptions are made about the nature of the asymptotic symmetry group—not even the assumption that such a group exists. One constructs the most sensible boundary conditions one can and then at the end investigates the question of asymptotic symmetries. The result is unexpected: Not only are the Lorentz transformations asymptotic symmetry transformations; there are also additional transformations which are *not* Lorentz transformations but *are* asymptotic symmetry transformations.[3,5,7] This paper consists of an attempt to understand this result.

The essential argument to be presented is that the actual asymptotic symmetry group is so similar to the Lorentz group that a surprisingly large number of our Lorentz covariant ideas remain applicable. A detailed discussion of the similarities is found in Secs. IV and V. Here we shall discuss only the two most important ones: (1) The asymptotic symmetry transformations actually form a group; the structure of this group does not depend on the particular gravitational field that happens to be present; (2) the four-parametric group of rigid space-time translations can be uniquely singled out from the entire asymptotic symmetry group by an invariant criterion.

The first result means that, in effect, one is able to separate the kinematics of space-time from the dynamics of the gravitational field at least at spatial infinity. This separation of kinematics and dynamics is so familiar in Lorentz covariant theories that its importance is easily overlooked. However, in general relativity such a separation is usually impossible. One is then faced by very puzzling questions, especially in a quantized theory. Should one, for example, regard coordinates as $c$ numbers or as operators?[9] Both alternatives seem unreasonable. By obtaining a partial separation of kinematics and dynamics one is able to avoid this and similar questions, at least for the time being; in particular, if the preferred coordinate transformation group is truly field-independent then one is presumably entitled to regard the preferred coordinates as $c$ numbers.

Unless the second result—the uniqueness of translations—held, one would have little hope of introducing reasonable energy-momentum conservation laws. In fact, the question of energy conservation which has plagued general relativity for many years is, of course, intimately related to the question of symmetries.[1] It seems to the author that it would be more accurate to claim that the notion of energy is simply not well defined in a general gravitational field than to claim that there are very small corrections to the energy due to the gravitational effects. Again, the most reasonable way out of the dilemma is to work with asymptotically flat space-times.[1]

Of course, one should also ask about the angular momentum conservation laws. Here there is a possibility that, even if the arguments for assuming that the asymptotic symmetry group is larger than the Lorentz group turn out to be correct, a theorem like IV.1 may avoid gross contradictions with microphysics.[10]

In Sec. II a series of known results needed in the remaining sections is collected. The basic ideas and definitions are introduced. The name "generalized Bondi-Metzner group" ("GBM" group) is suggested for the particular asymptotic symmetry group analyzed in this paper.

In Sec. III a new derivation of the asymptotic symmetry transformations is given. The discussion is less general than those previously given because it deals only with infinitesimal transformations. However, by using the methods of Newman and Unti,[7] one can avoid a rather dubious and controversial assumption made in the former discussions.

The main section, Sec. IV, opens with a discussion of a simple way to avoid the undesired extra asymptotic symmetry transformations; this can be achieved by setting additional boundary conditions at infinite times; the author does not think that this trick is a legitimate one according to our present knowledge of gravitational fields because it is based on assuming that the space is completely flat (or, say, static) at infinite times; but it may be that further investigations will validate the legitimacy of the trick.

A quite detailed analysis of the structure of the GBM group is then given, and two results on representations of the GBM group algebra are proved. Primary emphasis is placed on the similarities and differences between the GBM group and the inhomogeneous Lorentz group. Standard group theoretical techniques are used throughout Sec. IV, although there are some (rather trivial) difficulties that arise because the GBM group is not locally compact.

In any general relativistic discussion of asymptotically flat spaces one can give relatively precise treatments of otherwise inaccessible problems once one knows what the allowed coordinate transformations are.[3,5,7] Section V contains two applications of the group theoretical discussion given in Sec. IV. One can write down some GBM covariant quantum commutation rules for certain of the gravitational field variables. One can then use these commutators to guess at

and R. Sachs, Proc. Roy. Soc. (London) **264**, 309 (1961). A key paper is I. Robinson and A. Trautman, Proc. Roy. Soc. (London) **265**, 463 (1962).

[9] This point was raised by J. Anderson (private communication).

[10] Compare the arguments in P. G. Bergmann, Phys. Rev. **124**, 274 (1961).

integral invariants of the classical theory by looking for generators of GBM transformations in the quantum theory. In Sec. V a denumerably infinite set of integral invariants is obtained and their transformation properties in the classical theory are discussed in detail; four of the integral invariants represent the total energy and momentum carried to infinity by outgoing gravitational waves during the history of the field and transform as a Lorentz "free vector" under all GBM transformations.

In Sec. VI a brief discussion of the main results is given and three unsolved problems of interest are mentioned.

The detailed calculations in Secs. III, IV, and V are believed to be new. As emphasized in the acknowledgments, the essential ideas are based on previous papers and on discussions with others.

## II. PRELIMINARY CONSIDERATIONS

In this section a few known results of interest in the following discussions will be collected. Proofs will be omitted.

### A. Lorentz-Covariant Theories

Suppose one wants to analyze the solutions $\mu$ of D'Alembert's equation

$$\mu_{,ab}\eta^{ab}=0. \tag{II.1}$$

(Lower case Latin indices range and sum from zero to three; $\eta^{ab}$ is the Lorentz metric defined by the equations

$$ds^2=\eta_{ab}dx^adx^b=-dt^2+dx^2+dy^2+dz^2,\ \eta_{ab}\eta^{bc}=\delta_a{}^c; \tag{II.2}$$

commas denote ordinary derivatives.) It is then often convenient to introduce as new coordinates a retarded time $u$ and the spherical coordinates $r$, $\theta$, $\phi$:

$$u=t-r,\quad r\cos\theta=z,\quad r\sin\theta e^{i\phi}=x+iy. \tag{II.3}$$

In terms of the new coordinates the Minkowski metric takes the form

$$ds^2=-du^2-2dudr+r^2(d\theta^2+\sin^2\theta\, d\phi^2). \tag{II.4}$$

How do Lorentz transformations look in the new coordinate system? If one subjects the coordinates $x^a$ to a Lorentz transformation $x^a\rightarrow\bar{x}^a$ then Eq. (II.3) determines a corresponding transformation $(u,r,\theta,\phi)\rightarrow(\bar{u},\bar{r},\bar{\theta},\bar{\phi})$. The full transformation equations are somewhat complicated; however, it turns out that analyzing the transformation law of $u$, $\theta$, and $\phi$ at $r=\infty$ and discussing only those ("orthochronous") transformations that involve no time reversal is sufficient for our present purposes. Even at $r=\infty$ the two angles in general undergo what is called a "conformal transformation" and we must digress briefly to introduce a few relevant ideas and results.

Infinitesimal distances on the surface of the unit sphere are given by[11]

$$ds^2=d\theta^2+\sin^2\theta\, d\phi^2\equiv {}_0h_{AB}dx^Adx^B,\qquad (A,B,\cdots=2,3). \tag{II.5}$$

By a conformal transformation we mean any relation

$$\bar{\theta}=H(\theta,\phi),\quad \bar{\phi}=I(\theta,\phi), \tag{II.6}$$

for which

$$ds^2=K^2(\bar{\theta},\bar{\phi})(d\bar{\theta}^2+\sin^2\bar{\theta}\, d\bar{\phi}^2)$$

implying

$$K^4=\sin^2\theta\, J^2(\theta,\phi;\bar{\theta},\bar{\phi})(\sin\bar{\theta})^{-2}, \tag{II.7}$$

where $J$ is the Jacobian and

$$d\bar{\theta}\equiv(\partial H/\partial\theta)d\theta+(\partial H/\partial\phi)d\phi,$$
$$d\bar{\phi}\equiv(\partial I/\partial\theta)d\theta+(\partial I/\partial\phi)d\phi.$$

As a convention we choose $K$ positive. It is known[12] that conformal transformations in two dimensions correspond to analytic transformations of a complex variable $z$; here

$$z=\cot(\theta/2)e^{i\phi}. \tag{II.8}$$

In order for the transformation (II.6) to be regular everywhere on the sphere the corresponding transformation of $z$ must be single valued, have at most one simple pole (at the new North Pole), and have at most one simple zero (at the new South Pole). Therefore, if $\bar{z}$ denotes the transformed $z$, one has

$$\bar{z}=(\delta z+\eta)/(\beta z+\tau),\quad \delta\tau-\eta\beta\neq 0, \tag{II.9}$$

where $\delta$, $\eta$, $\beta$, and $\tau$ are complex parameters. Without loss of generality one can demand

$$\delta\tau-\eta\beta=\pm 1. \tag{II.10}$$

And the composition law also turns out to be the composition law for unimodular matrices. Therefore,[13] *the conformal transformations of the unit sphere into itself form a six-parametric group isomorphic to the homogeneous orthochronous (improper) Lorentz group.*

One can now state the asymptotic result obtained when the coordinates are subjected to a Lorentz transformation. *To every orthochronous Lorentz transformation corresponds precisely one transformation*

$$\begin{aligned}&\lim\bar{\theta}=H(\theta,\phi),\\&\lim\bar{\phi}=I(\theta,\phi),\\&\lim\bar{u}=K^{-1}(u+\epsilon_0+\epsilon_1\sin\theta\cos\phi+\epsilon_2\sin\theta\sin\phi+\epsilon_3\cos\theta),\end{aligned} \tag{II.11}$$

*where $H$, $I$, and $K$ are the functions of a conformal transformation and $\epsilon_a$ $(a=0\cdots 3)$ are four parameters;*

[11] The second equality merely serves to define ${}_0h_{AB}$; the range (2,3) for capital Latin letters is chosen for later convenience.

[12] The various geometrical concepts here introduced are discussed in L. P. Eisenhart, *Riemannian Geometry* (Princeton University Press, Princeton, New Jersey, 1949); and in J. A. Schouten, *Ricci Calculus* (Springer-Verlag, Berlin, 1959).

[13] P. Roman, *Theory of Elementary Particles* (North-Holland Publishing Company, Amsterdam, 1960).

*conversely, to each set* $(H,I,\epsilon_a)$ *corresponds precisely one orthochronous Lorentz transformation.*[5]

Here, and throughout, the symbol "lim" means the limit as $r$ approaches infinity with fixed values of $u$, $\theta$, and $\phi$. The parameters $\epsilon_a$ correspond to the translations. For example,

$$\lim\bar{\theta}=\theta,\quad \lim\bar{\phi}=\phi,\quad \lim\bar{u}=u-\epsilon_3\cos\theta \qquad \text{(II.12)}$$

determines a rigid translation in the $z$ direction. The transformation law for $r$ is $\lim(\bar{r}/r)=K(\theta,\phi)$.

### B. Uniform Smoothness and the Outgoing Radiation Condition

Suppose one has a solution of Eq. (II.1) which is everywhere nonsingular and consists merely of a burst of radiation that comes in from spatial infinity, passes through the origin, and travels back out to spatial infinity. The two solutions

$$\mu_1=a(u)/r-a(u+2r)/r,$$

(where $a$ is any function that vanishes outside a bounded interval),

$$\mu_2=(\sin u)e^{-u^2}/r-\sin(u+2r)e^{-(u+2r)^2}/r \qquad \text{(II.13)}$$

are typical. Each of these solutions is $O(r^{-1})$ as $r\rightarrow\infty$ for fixed $u$, $\theta$, and $\phi$, where the order symbol "$O$" has its usual meaning. Moreover, the derivatives obey

$$\begin{aligned}\partial\mu_{1,2}/\partial u&=O(r^{-1}), & \partial\mu_{1,2}/\partial\theta&=O(r^{-1}),\\ \partial\mu_{1,2}/\partial r&=O(r^{-2}), & \partial\mu_{1,2}/\partial\phi&=O(r^{-1}).\end{aligned} \qquad \text{(II.14)}$$

*Definition*[6]: A function $\mu=O(r^{-N})$ of $u$, $r$, $\theta$, and $\phi$ is called "uniformly and radially smooth (at infinity)" if

$$\begin{aligned}\partial\mu/\partial r&=O(r^{-N-1}), & \partial\mu/\partial\theta&=O(r^{-N}),\\ \partial\mu/\partial u&=O(r^{-N}), & \partial\mu/\partial\phi&=O(r^{-N}).\end{aligned} \qquad \text{(II.15)}$$

Not all solutions of Eq. (II.1) are uniformly and radially smooth. Consider, for example, the solution

$$\mu=\sin u/r-\sin(u+2r)/r; \qquad \text{(II.16)}$$

one sees that

$$\partial\mu/\partial r=O(r^{-1}) \qquad \text{(II.17)}$$

so the radial smoothness assumption is violated. Why this difference?

It is easily seen that for fields which approach zero at $r=\infty$ the behavior as regards uniform smoothness depends entirely on the Sommerfeld outgoing radiation condition. In fact, the two solutions (II.13) both obey the Sommerfeld condition in the following sense: For fixed $u$, $\theta$, and $\phi$ the outgoing radiation condition is obeyed at large $r$; for fixed $\theta$, $\phi$, and $v=u+2r$ the *incoming* radiation condition is obeyed for large $r$. On the other hand the solution (II.16) does not obey either the outgoing or incoming radiation condition for any $r$. Various recent papers[1,2,5,14] attempt to translate the above simple comments about the Sommerfeld condition into a form suitable for gravitational theory.

In gravitational theory it seems to be unreasonable to consider solutions analogous to (II.16) because these solutions carry, crudely speaking, an infinite amount of energy in the radiation modes of the field; this energy apparently gives rise to an infinite mass term in the field. In this sense the assumption of asymptotic flatness (which certainly excludes an infinite mass term) may by itslf imply a Sommerfeld-type condition; if correct, this statement indicates a marked difference between gravitational theory and linear theories.

### C. The Gravitational Case—Local Considerations

In analyzing gravitational fields it is sometimes useful to introduce coordinates which share some of the properties of the coordinates $u$, $r$, $\theta$, and $\phi$. The crucial properties are: (i) the hypersurfaces $u=$constant are everywhere tangent to the local lightcone; (ii) $r$ is the corresponding luminosity distance; (iii) the scalars $\theta$ and $\phi$ are constant along each "ray." A ray is defined as the line with tangent $k^a=-u_{,b}g^{ab}$, where $g^{ab}$ is the metric tensor.

*Given any normal-hyperbolic Riemannian manifold with line element $ds^2$ and in it any point $P$, there exists at least one set of coordinates $u=x^0$, $t=x^1$, $\theta=x^2$, $\phi=x^3$ such that in a finite neighborhood of $P$*

$$\begin{aligned}ds^2=e^{2\beta}Vr^{-1}du^2-2e^{2\beta}dudr&\\ +r^2h_{AB}(dx^A-U^Adu)(dx^B-U^Bdu),&\end{aligned} \qquad \text{(II.18)}$$

where

$$A,\,B,\,\cdots=2,\,3;\quad \text{determinant}(h_{AB})=b(u,\theta,\phi).$$

*The form* (II.18) *holds if and only if the coordinates $u$, $r$, $\theta$, and $\phi$ have the geometric properties* (i)–(iii) *stated above.* Here $V$, $\beta$, $U^A$, and $h_{AB}[\text{determinant}(h_{AB})]^{-(1/2)}$ are any six functions of the coordinates and $b(u,\theta,\phi)$ is any function of its arguments.

The proof was given in reference 5; rather than repeat the proof we repeat here a diagram (Fig. 1) from that reference which shows the geometric properties of the coordinates in the generic case.

### D. Gravitational Case—Global Considerations

In references 3 and 5 a more interesting, global problem was attacked. It was assumed that there was some global coordinate system in which the metric takes the form (II.18). The convention $b=\sin^2\theta$ was made and it was assumed that the coordinate ranges in which the form (II.18) holds are

$$u_0\leq u\leq u_1,\quad r_0\leq r\leq\infty,\quad 0\leq\theta\leq\pi,\quad 0\leq\phi\leq 2\pi \qquad \text{(II.19)}$$

(compare Fig. 1). It was further assumed that the metric approaches the Minkowski metric for large $r$:

$$\lim(ds^2)=-du^2-2dudr+r^2(d\theta^2+\sin^2\theta\,d\phi^2). \qquad \text{(II.20)}$$

[14] R. Sachs, reference 8.

And, finally, an outgoing radiation condition was assumed. In references 6 and 7 these assumptions were generalized by considering relations valid over different ranges for $\theta$ and $\phi$ and weakening the outgoing radiation condition. By these manipulations one tried to analyze the set of all gravitational fields which are asymptotically flat at infinity. The author feels that the work of these references is not as rigorous or clear as one would like[15]; he also feels that this work represents a serious attempt to handle the main problems of gravitational theory using the physical and mathematical knowledge available at present.

By analyzing the field equations it was shown in the references[3,5,7] that the detailed asymptotic behavior of the quantities in (II.18) is the following:

$$\begin{aligned} V&=-r+2M(u,\theta,\phi)+O(r^{-1}),\\ \beta&=-c(u,\theta,\phi)c^*(2r)^{-2}+O(r^{-4}),\\ h_{AB}dx^Adx^B&=(d\theta^2+\sin^2\theta d\phi^2)+O(r^{-1}),\\ U^A&=O(r^{-2}). \end{aligned} \tag{II.21}$$

$c^*$ is the complex conjugate of $c$.

*Definition.* A space-time is said to be an "AF field" if there exists at least one set of coordinates $(u,r,\theta,\phi)$ for which Eqs. (II.18), (II.19), and (II.21) hold. If the order symbols in Eq. (II.21) are, together with their first $N-1$ derivatives, uniformly and radially smooth the AF field is further said to be "uniformly and radially smooth of order $N$ (at infinity)."

All our subsequent discussions are confined to such space times.

### E. The Generalized Bondi-Metzner Group

Bondi and Metzner[3] analyzed the set of all coordinate transformations which preserves the form (II.18), (II.19), (II.21) in the case that the field is axially and reflection symmetric (technically, the $\phi$ directions form a congruence of closed, spacelike, Killing curves which are orthogonal to hypersurfaces). Their considerations were subsequently generalized[5,7] and the following group obtained.

Suppose one has a space smoothly covered by the three coordinates $\theta$, $\phi$, and $u$, where $0\leq\theta\leq\pi$, $0\leq\phi\leq 2\pi$, $-\infty\leq u\leq\infty$; the points $\phi=0$ are identical to the points $\phi=2\pi$. The space is to be topologically equivalent to the topological product of the real axis with the surface of the unit two-dimensional sphere. Consider a transformation

$$\bar\theta=\bar\theta(\theta,\phi,u),\quad \bar\phi=\bar\phi(\theta,\phi,u),\quad \bar u=\bar u(\theta,\phi,u), \tag{II.22}$$

[15] Specifically, our following discussion is not very interesting unless all spaces that could reasonably be called asymptotically flat are actually AF spaces in the sense of the definition given below; that such is the case has been made quite plausible but not really proved; in order to avoid prejudicing this important point the neutral term "AF field" is used rather than the more vivid term "asymptotically flat field."

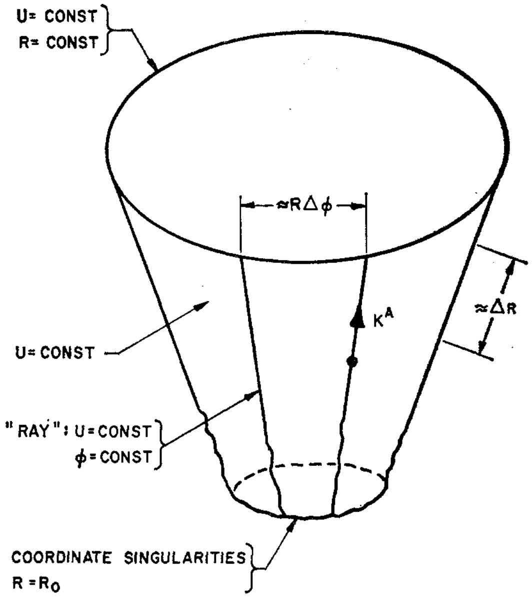


FIG. 1. The coordinate system. The coordinates $u$, $r$, and $\phi$ and the vector $k^a$ shown in the hypersurface $\theta=\pi/2$. In Sec. II(c) we consider a small patch cut from the picture; thereafter, the global picture is appropriate.

to another set of coordinates with the same coordinate ranges.

*Definition.* A transformation $(u,\theta,\phi)\rightarrow(\bar u,\bar\theta,\bar\phi)$ is called a generalized Bondi-Mentzner transformation ("GBM transformation") if

$$\bar\theta=H(\theta,\phi),\quad \bar\phi=I(\theta,\phi),\quad \bar u=K^{-1}[u+\alpha(\theta,\phi)], \tag{II.23}$$

where $H$, $I$, and $K$ are the functions of a conformal transformation (II.6), (II.7), and $\alpha$ is any twice differentiable function of $\theta$ and $\phi$.

*The GBM transformations form a group.* In fact, the conformal transformations form a group, so that $H$, $I$, and $K$ have all the necessary properties. Thus, one must check only the fact that if one carries out two transformations (II.23) in succession, the corresponding $\alpha$ for the product transformation is again a twice differentiable function of $\theta$ and $\phi$. This fact can be verified by noting that any transformation (II.6), (II.7), can be written as the product of "spatial transformations," characterized by the relation $K=1$, with the special Lorentz space-time rotation (in the $z$, $t$ plane)

$$\cot(\bar\theta/2)=\delta^2\cot(\theta/2),\quad \bar\phi=\phi,\quad \bar u=u, \tag{II.24}$$

where $\delta$ is a real constant.[16] Both the spatial rotations and the transformations (II.24) lead from twice differentiable functions $\alpha$ to other twice differentiable functions $\alpha$. Q.E.D.

[16] S. Schweber, *Relativistic Quantum Field Theory* (Row-Peterson and Company, Evanston, Illinois, 1961).

### F. Isometries

In the next section we will give a detailed discussion of the relation between the GBM group and the asymptotic symmetries of AF fields. This preliminary section will be concluded by a discussion of the geometrical notion of symmetry in a curved space time. Suppose one has the metric tensor $g_{ab}$ as a function of coordinates $x^a$. Suppose that there exists a one-parametric set of transformations $\bar{x}^a=\bar{x}^a(\omega;x^b)$, $\bar{x}^a(0;x^b)=x^a$, which are such that the transformed metric is the same function of the new coordinates as $g_{ab}$ is of the $x^a$. Then the transformations are called symmetry or "isometry" transformations. Let $\xi^a=[\partial\bar{x}^a/\partial\omega]_{\omega=0}$; $\xi^a$ is a contravariant vector. The basic way to look for isometries in a given space time is to use the following well-known theorem: *The vector $\xi^a$ obeys "Killing's equation"*

$$\xi_{a;b}+\xi_{b;a}=0 \tag{II.25}$$

*if, and only if, the corresponding transformations are isometries.*[12] Here and throughout semicolons denote covariant derivatives.

By means of specific examples one can convince oneself that any reasonably general space-time permits either no solutions of Killing's equation or at most one solution of Killing's equation. To look for general, physically realistic properties of gravitational radiation from bounded sources in a space-time that permits two linearly independent solutions of Killing's equation is not a sensible procedure. Minkowski space, of course, permits ten linearly independent solutions, corresponding to the ten-parametric Lorentz group.

## III. INFINITESIMAL ASYMPTOTIC SYMMETRY TRANSFORMATIONS

The proofs given in the references[3,5] that the GBM group is that group which preserves the boundary conditions (II.21) all contain one *ad hoc* assumption. It is assumed that the transformation functions can be expanded in inverse powers of the luminosity distance $r$ over the coordinate range (II.19). Thus, the transformations $\bar{r}=r+\ln r+O(r^{-1})$ and all similar transformations are banished *a priori*. This would be merely a point of technical rigor were it not for the fact that in a closely related paper by Bergmann[10] it is precisely such logarithmic transformations which play a crucial role. It is desirable to remove the *ad hoc* assumption mentioned. In this section the required theorem will be proved for the case of infinitesimal transformations; the proof exhibits particularly clearly what is meant by the words "asymptotic symmetry." It is a straightforward extension of calculations by Unti and Newman.[7]

Let $\bar{x}^a=\bar{x}^a(\omega;x^b)$ where $x^0=u$, $x^1=r$, etc. Let $\bar{x}^a(0;x^b)=x^a$ and let $\xi^a$ be defined as in Sec. II F. Then the infinitesimal change $\bar{\delta}g_{ab}$ of the metric tensor as a function of its arguments is, as is well known, given by the Lie derivative of the metric in the $\xi^a$ direction; that is,[12]

$$\bar{\delta}g_{ab}=-\xi_{a;b}-\xi_{b;a}. \tag{III.1}$$

Suppose now, that one is dealing with an AF field (which is here assumed uniformly and radially smooth of order 3 at spatial infinity). By virtue of Eqs. (II.18) and (II.21) one must demand the following conditions in order for the coordinate conventions and boundary conditions to remain invariant:

$$\bar{\delta}g_{11}=0,\quad \bar{\delta}g_{1A}=0,\quad \bar{\delta}g_{AB}g^{AB}=0; \tag{III.2}$$

$$\begin{aligned}\bar{\delta}g_{00}&=O(r^{-1}),\quad \bar{\delta}g_{0A}=O(1),\\ \bar{\delta}g_{01}&=O(r^{-2}),\quad \bar{\delta}g_{AB}=O(r).\end{aligned} \tag{III.3}$$

*Theorem* III.1. *The infinitesimal asymptotic symmetry transformations* (III.1)–(III.3) *form an infinitesimal group isomorphic to the infinitesimal* GBM *group.*

*Proof.* The crucial point is that Eqs. (III.2) can be explicitly integrated to find the functional dependence of $\xi^a$ on $r$. One then sees explicitly that $\xi^a$ does not contain any logarithmic terms (or any terms which are not uniformly and radially smooth) unless the metric itself contains such terms; in this way one proves that $\xi^a$ is sufficiently well behaved for the former proofs to be applicable.

The reader who wishes to follow the proof in detail will find it useful to consider only the axially symmetric case treated by Bondi, Van der Burg, and Metzner; the Christoffel symbols for this case have been given[3] (while the general case has been investigated only in the tetrad formalism); moreover, as frequently happens, considering non-axially symmetric fields merely leads to extra calculations without any new ideas appearing.

From the first equation in (III.2) one finds that

$$\partial\xi_1/\partial r=2\xi_1\partial\beta/\partial r. \tag{III.4}$$

Therefore,

$$\xi_1=f(u,\theta,\phi)e^{2\beta} \tag{III.5}$$

where $f$ is an arbitrary function of its arguments. The next two equations in (III.2) similarly imply

$$\xi_B g^{AB}=f^A(u,\theta,\phi)+fU^A+\int_r^\infty dr'\, e^{2\beta}f_{,B}g^{AB}, \tag{III.6}$$

where the functions $f^A$ are arbitrary. The last equation in (III.2) can be solved algebraically for $\xi_0$ as follows:

$$\xi_0=-\tfrac{1}{2}re^{2\beta}(-\xi_{A,B}+\xi_C\Gamma_{AB}{}^C+\xi_1\Gamma_{AB}{}^1)g^{AB}.$$

Here $\Gamma_{AB}{}^C$ and $\Gamma_{AB}{}^1$ are Christoffel symbols in the original coordinate system.

The remainder of the proof is quite trivial. One substitutes into the remaining six equations (III.3) and finds that Eq. (III.3) holds if and only if the previously arbitrary functions $f$ and $f^A$ are now restricted by the relations

$$\text{(a)}\qquad \partial f^A/\partial u=0, \tag{III.8}$$

$$\text{(b)}\quad f_{A:B}+f_{B:A}=-2(\partial f/\partial u)_0 h_{AB}\Rightarrow \partial^2 f/\partial u^2=0.$$

Here the indices of $f^A$ are lowered with the metric ${}_0h_{AB}$ of the unit sphere (II.5) and the colon derivates denote covariant derivatives with respect to this metric (in other words, since $f^A$ is independent of $r$ one must work merely with the limiting form $g_{AB}\rightarrow {}_0h_{AB}$). Equation (III.8b) is the well-known equation for an infinitesimal conformal transformation.[12] Therefore, Eqs. (III.5)–(III.8) are precisely the infinitesimal analogs of Eq. (II.23), which defines the GBM group; Q.E.D.

Equations (III.1), (III.2), and (III.3) should be compared with Eq. (II.25). One sees that, while $\xi^a$ is not a solution of Killing's equation, it is as close to being an isometry as the curvature of the space-time allows. Thus, in the particular case under consideration here, Eqs. (III.1), (III.2), and (III.3) provide an unambiguous and detailed definition of the notion of asymptotic symmetry.[17]

## IV. THE STRUCTURE OF THE GENERALIZED BONDI-METZNER GROUP

### A. General Discussion

The results given up to this point are quite disconcerting. We wanted to find the inhomogeneous Lorentz group as an asymptotic symmetry group; instead we obtained the GBM group. One way out is to assume that the space is completely flat for a semi-infinite retarded time interval $u_1\leq u\leq\infty$. Then one simply insists that the metric take the form (II.4) at $u=\infty$ and all $r$. Since the function $\alpha$, which causes all the trouble, is time independent this additional convention can easily be shown to lead right back to the inhomogeneous orthochronous Lorentz group.[18] But the assumption that the space is strictly flat at $u=+\infty$ is rather *ad hoc* and may be unrealistic. The object of this section is to argue that there is also a subtler way out of the difficulty. One retains the GBM group, and then finds that it has so many simple properties that an indirect kind of Lorentz covariance can still be said to hold.

The most important result is that there is a unique way to identify the four rigid displacements in space-time. Only if one has the translations defined can one hope to get a reasonable Hamiltonian theory or sensible energy-momentum conservation laws. Another rather comforting result is that at least one irreducible Hermitian representation of the infinitesimal inhomogeneous Lorentz group algebra corresponds in a natural way to a similar representation of the infinitesimal GBM group algebra. Moreover, as will be seen, the group theoretical notion of a rest mass operator can be retained without essential modifications. However, there are also important differences; in particular, the spin operator is not an invariant of the group representations.

It will be useful to introduce special names for some of the subgroups of the GBM group. The transformations (II.23) for which $J$ defined in Eq. (II.7) is positive form the subgroup of "proper" GBM transformations. We have already discussed the "conformal" subgroup $L$ obtained by setting $\alpha=0$ in Eq. (II.23). The transformations

$$\bar{\theta}=\theta,\quad \bar{\phi}=\phi,\quad \bar{u}=u+\alpha \tag{IV.1}$$

form the "supertranslation" subgroup $N$. The supertranslations for which

$$\alpha=\epsilon_0+\epsilon_1\sin\theta\cos\phi+\epsilon_2\sin\theta\sin\phi+\epsilon_3\cos\theta \tag{IV.2}$$

form the "translation" subgroup.

### B. Normal Subgroups

An important question about any group is the question of what normal subgroups the group contains; a normal subgroup $N$ of any group $G$ is characterized by the property that if $n$ is an element of $N$ and $g$ an element of $G$ then the "commutator" $g^{-1}n^{-1}gn=n'$ is an element of $N$.[12,19]

*Theorem* IV.1. *The supertranslations form an Abelian normal subgroup $N$ of the generalized Bondi Metzner group; the factor group is isomorphic to the orthochronous homogeneous Lorentz group.*

*Proof*. The supertranslations are characterized by the fact that they leave the angles $\theta$ and $\phi$ unchanged. Using this fact, one finds that the supertranslations form a normal subgroup. The factor group is obviously isomorphic to the conformal subgroup $L$ defined above. In Sec. II we have shown that this conformal subgroup is isomorphic to the homogeneous orthochronous Lorentz group; that any two supertranslations commute follows from Eq. (II.23), Q.E.D.

*Lemma 1. The translations form a normal four-dimensional subgroup of the proper Bondi-Metzner group.* In fact, any translation commutes with any supertranslation. As can be seen from the Lorentz group the commutator of a translation with a conformal transformation is some translation. Therefore, the translations form a normal subgroup. This normal subgroup is four dimensional since it requires, as one sees from Eq. (IV.2), exactly four parameters to span the translation group, Q.E.D.

*Lemma 2. If $N'$ is a four-dimensional normal subgroup of the proper* GBM *group then $N'$ is contained in the supertranslation group $N$.* In fact, let $G$ be the proper GBM group and consider the image $N'/N$ of $N'$ under

[17] A quite similar calculation has recently been carried through by A. Komar, Syracuse University. The author is grateful to Dr. Komar for making his results available prior to publication.

[18] A similar conclusion was previously and independently reached by R. Arnowitt, S. Deser, and C. W. Misner [C. W. Misner (private communication)].

[19] A few essential ideas about group theory will be reviewed here; for a complete treatment many excellent texts can be consulted, for example, L. Pontrjagin, *Topological Groups* (Princeton University Press, Princeton, New Jersey, 1946); or N. Bourbaki, *Groupes et Algebres de Lie* (Hermann & Cie, Paris, 1960), Vol. 26, p. 1.

the homomorphism $G \to G/N$; since $N'$ is a normal subgroup of $G$, $N'/N$ is a normal subgroup of $G/N$. Therefore, according to theorem IV.1, $N'/N$ is a normal subgroup of the proper orthochronous homogeneous Lorentz group $L'$. However, the only normal subgroups of $L'$ are $L'$ itself and the identity "$e$" of $L'$. If $N'/N=L'$ then $N'/N$ must be six dimensional; then $N'$ is at least six dimensional, contrary to hypothesis. Therefore, $N'/N=e$; $N'$ is, therefore, contained in $N$, Q.E.D.

### C. Infinitesimal Transformations

To complete the discussion, one must introduce the infinitesimal GBM transformations.[20] Suppose one has any $S$-dimensional Lie transformation group of an $R$-dimensional space. Let the coordinates of the space be $y^\alpha$ $(\alpha,\beta=1\cdots R)$ and the parameters of the group be $z^\mu$ $(\gamma,\mu=1\cdots S)$, where $z^\mu=0$ is the identity of the group. Then the transformations have the form

$$\bar{y}^\alpha=f^\alpha(y^\beta;z^\mu),\quad \text{where}\quad f^\alpha(y^\beta;0)=y^\alpha; \qquad \text{(IV.3)}$$

the functions $f^\alpha$ are assumed to be twice differentiable. Consider the quantities

$$q_\mu{}^\alpha=(\partial f^\alpha/\partial z^\mu)_{z^\mu=0},\quad (\alpha=1\cdots R;\mu=1\cdots S). \qquad \text{(IV.4)}$$

$q_1{}^\alpha$ is a vector defined everywhere on the $R$-dimensional space; there are $S$ such vectors, one for each of the group parameters. If the group is truly $S$ dimensional and the $z^\mu$ are chosen suitably, these vectors are linearly independent; that is, if one has $S$ constants $B^\mu$ then

$$B^\mu q_\mu{}^\alpha=0 \Rightarrow B^\mu=0,\quad (\alpha=1\cdots R;\mu=1\cdots S). \qquad \text{(IV.5)}$$

To investigate the group structure one introduces the differential operators $P_\mu$:

$$P_\mu=q_\mu{}^\alpha(\partial/\partial y^\alpha),\quad (\alpha=1\cdots R;6=1\cdots S); \qquad \text{(IV.6)}$$

these are again linearly independent in the above sense. Consider a sufficiently small, finite neighborhood of the identity $z^\mu=0$ in the space of different group elements. Every group element in such a neighborhood lies in precisely one one-dimensional subgroup of the full group. To each one-dimensional subgroup corresponds precisely one set $\{Q'\}$ of multiples of a particular linear combination of the basic linear operators

$$Q=B^\mu P_\mu,\quad Q'=BQ. \qquad \text{(IV.7)}$$

Here the $B^\mu$ are again constants and $B$ is an arbitrary constant. Conversely, to every set $\{Q'\}$, defined by Eq. (IV.7) with fixed $B^\mu$ and arbitrary $B$ corresponds precisely one one-dimensional subgroup of the group.

To the commutator of two group elements $g$ and $h$ that lie in the relevant neighborhood of the group identity corresponds the Lie commutator

$$(G,H)=-(H,G)=G^\mu H^\gamma(P_\mu,P_\gamma)$$
$$=(G^\mu H^\gamma-H^\mu G^\gamma)(q_\mu{}^\alpha\partial q_\gamma{}^\beta/\partial y^\alpha)(\partial/\partial y^\beta), \qquad \text{(IV.8)}$$
$$G=G^\mu P_\mu,\quad H=H^\mu P_\mu,\quad (\alpha,\beta=1\cdots R;\mu,\gamma=1\cdots S),$$

of differential operators $G$ and $H$ that correspond to the subgroups in which $g$ and $h$ lie. The group axioms imply that the Lie commutator must be a linear combination of the basic differential operators:

$$(P_\mu,P_\gamma)=A_{\mu\gamma}{}^\rho P_\rho,\quad (\rho,\mu,\gamma=1\cdots S). \qquad \text{(IV.9)}$$

Here the quantities $A_{\mu\gamma}{}^\rho$ are constants, known as the structure constants. The differential operators, considered as abstract quantities whose only relevant properties are given by their commutator table, are a complete linearly independent set of basis elements for what is called the "Lie algebra" of the group. The Lie algebra itself consists of the linear combinations of the differential operators; commutators are imposed on these linear combinations in the obvious way and then obey all the usual abstract properties of Poisson or commutator brackets, such as antisymmetry and bilinearity.[19]

To an $S'<S$ dimensional subgroup correspond $S'$ linearly independent operators $Q_\mu$ (in this subsection the symbol $B$ with indices will denote constants):

$$Q_\mu=B_\mu{}^\gamma P_\gamma,\quad (\mu=1\cdots S';\gamma=1\cdots S;S'<S). \qquad \text{(IV.10)}$$

These operators have the property that their Lie commutators are linear combinations of themselves:

$$(Q_\mu,Q_\rho)=B_{\mu\rho}{}^\gamma Q_\gamma,\quad (\gamma,\mu,\rho=1\cdots S'). \qquad \text{(IV.11)}$$

If the $S'$ dimensional subgroup is a normal subgroup then the $Q_\mu$ obey the stronger conditions

$$(Q_\mu,P_\gamma)=Q_\rho B_{\mu\gamma}{}'^\rho,\quad (\rho,\mu=1\cdots S';\gamma=1\cdots S). \qquad \text{(IV.12)}$$

The fundamental theorem on Lie groups states that these relations can be inverted: If one can find in the Lie algebra combinations $Q_\mu$ that obey Eqs. (IV.11) or (IV.12), then there exists, respectively, a subgroup or normal subgroup to which the $Q_\mu$ correspond. Thus the structure of the Lie algebra characterizes the structure of the Lie group up to those global properties that cannot be analyzed by analyzing a small finite neighborhood of the group identity.

To apply these ideas to the GBM group let us expand $\alpha$ in spherical harmonics[21]:

$$\alpha=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} z_{lm}Y_{lm}(\theta,\phi),\quad z_{l-m}=z^*{}_{lm}. \qquad \text{(IV.13)}$$

From Eq. (IV.6) one finds for the supertranslations

$$P_{lm}=Y_{lm}(\theta,\phi)(\partial/\partial u),$$
$$(P_{lm}=P_{l-m}{}^*,\ P_{lm}=0 \text{ for } |m|>1). \qquad \text{(IV.14)}$$

[20] The GBM group is not locally compact (see reference 19). Therefore, one must be somewhat careful in trying to deduce properties of the finite group from properties of the infinitesimal group. In the proof of Theorem IV.2, following, no questionable deductions are necessary.

[21] Such an expansion is always possible, since the function $\alpha$ is twice differentiable; R. Courant and D. Hilbert, *Methoden d. Mathematischen Physik* (Verlag Julius Springer, Berlin, 1937), Vol. 1.

The six differential operators corresponding to the proper conformal group $L$ will be written $L^{ab}$, with $L^{ab}=-L^{ab}$. $L^{ab}$ generates an infinitesimal rotation within the $(x^a,x^b)$ plane of Minkowski space. For example, a rotation in the $(x,y)$ plane is given by

$$\bar\theta=\theta,\quad \bar\phi=\phi+\text{const},\quad \bar u=u; \tag{IV.15}$$

a rotation in the $(z,t)$ plane is given by

$$\cot(\bar\theta/2)=(1-\text{const})\cot(\theta/2),\qquad \bar u=K^{-1}(\theta)u,\quad \bar\phi=\phi. \tag{IV.16}$$

Therefore, Eqs. (IV.4) and (IV.6) give

$$L^{12}=(\partial/\partial\phi),\quad L^{30}=\sin\theta(\partial/\partial\theta)+u\cos\theta(\partial/\partial u). \tag{IV.17}$$

It is often convenient to introduce the following linear combinations:

$$\begin{aligned} &L_z=L^{12},\quad R_z=L^{30},\\ &L^{\pm}=\pm iL^{23}+L^{13}=e^{\pm i\phi}[\partial/\partial\theta\pm i\cot(\partial/\partial\phi)],\\ &-R^{\pm}=\pm iL^{20}-L^{10}=e^{\pm i\phi}[\cos\theta(\partial/\partial\theta)\\ &\qquad \pm i\csc\theta(\partial/\partial\phi)-u\sin\theta(\partial/\partial u)]. \end{aligned} \tag{IV.18}$$

The $P_{lm}$ and $L^{ab}$ together form a complete set of linearly independent differential operators for the infinitesimal GBM group. From Eq. (IV.18) one finds the basic commutators

$$\begin{aligned} &(L^{ab},L^{cd})=\eta^{ad}L^{bc}+\eta^{bc}L^{ad}-\eta^{ac}L^{bd}-\eta^{bd}L^{ac};\\ &(L^{ab},\alpha\partial/\partial u)=[(L^{ab}\alpha)-\alpha(\theta,\phi)W(L^{ab})]\partial/\partial u \end{aligned}$$

implying

$$\begin{aligned} &(L_z,P_{lm})=imP_{lm},\\ &(L^+,P_{lm})=-[(l-m)(l+m+1)]^{(1/2)}P_{l,m+1},\\ &(R_z,P_{lm})=(l-1)[(l-m+1)(l+m+1)]^{1/2}\\ &\qquad\times[(2l+1)(2l+3)]^{-1/2}P_{l+1,m}\\ &\qquad-(l+2)(l^2-m^2)^{1/2}(4l^2-1)^{-1/2}P_{l-1,m},\\ &(R^+,P_{lm})=(l-1)[(l+m+2)(l+m+1)]^{1/2}\\ &\qquad\times[(2l+1)(2l+3)]^{-1/2}P_{l+1,m+1}\\ &\qquad-(l+2m)[(l-m)(l-m-1)]^{1/2}\\ &\qquad\times(4l^2-1)^{-1/2}P_{l-1,m-1}. \end{aligned} \tag{IV.19}$$

Here $W(L^{ab})$ is defined by the relation $\partial(L^{ab}f)/\partial u \equiv L^{ab}\partial f/\partial u+W\partial f/\partial u$ for arbitrary $f(u)$. All other commutators can be obtained from those given in Eq. (IV.19) by taking linear combinations or complex conjugates, for example,

$$\begin{aligned} &(L_z,R_z)=(L^+,R^+)=0,\\ &(L_z,L^+)=i(R_z,R^+)=iL^+,\\ &(L_z,R^+)=-i(L^+,R_z)=iR^+,\\ &(L^+,L^-)=-(R^+,R^-)=2iL_z,\\ &(L^+,R^-)=2R_z;\\ &(L^-,P_{lm})\equiv(L^+,P_{lm}{}^*)^*=(L^+,P_{l,-m})^*. \end{aligned} \tag{IV.20}$$

From the last four relations in Eq. (IV.19) one sees that the conformal transformations transform the translations only among themselves (because of the factors $l-1$), but completely mix up all the other supertranslations with each other. This fact can be used to prove the uniqueness of the translation group.

*Theorem* IV.2. *The only normal four-dimensional subgroup of the GBM group is the translation group.*

In fact, suppose there were a second four-dimensional normal subgroup. Label the four linearly independent differential operators that correspond to the supposed second group as $P_a$ $(a=0\ldots3)$. Then from lemma 2 above one infers

$$P_a=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_a{}^{lm}P_{lm}. \tag{IV.21}$$

Moreover, there must be at least one value of $m$, one value of "$a$," and one value of $l\geq2$ for which $B_a{}^{lm}\neq0$, since the four linearly independent operators $P_{lm}$ with $l<2$ are merely the four operators of the translation group. Now let $n\geq2$ be the minimum value of $l\geq2$ for which there is at least one $B^{nm}{}_a\neq0$. Now choose one value of "$a$" for which $B_a{}^{nm}\neq0$. Let $m_0$ be the minimum value of $m$ for which $B_a{}^{nm}\neq0$ with this choice of "$a$." Now commute $P_a$ $(n-m_0)$ times with the operator $L^+$. The resulting operator $Q$ has the form

$$Q=BP_{nn}+\textstyle\sum_{l\neq n}\sum_m B^{lm}P_{lm},\qquad B\neq0,\quad \text{no sum on "}n\text{,"} \tag{IV.22}$$

as one sees from the commutator table (IV.19). Commute $Q$ four times successively with the operator $L^-$; one obtains

$$\begin{aligned} &(L^-,Q)\equiv Q'=B'P_{n,n-1}+\cdots,\quad (n\geq2;\,B'\neq0),\\ &(L^-,Q')\equiv Q''=B''P_{n,n-2}+\cdots,\quad (B''\neq0),\\ &(L^-,Q'')\equiv Q'''=B'''P_{n,n-3}+\cdots,\quad (B'''\neq0),\\ &(L^-,Q''')=B^{\text{iv}}P_{n,n-4}+\cdots,\quad (B^{\text{iv}}\neq0). \end{aligned} \tag{IV.23}$$

Now the five operators shown in Eqs. (IV.23) and (IV.24) are all linearly independent, since the $P_{nm}$ appearing in them are linearly independent. On the other hand, by virtue of Eq. (IV.12) and the hypothesis that we are dealing with a normal four-dimensional subgroup, these five operators must depend linearly on the original four $P_a$ (with complex coefficients). This is a contradiction and establishes the theorem, Q.E.D.

As desired, the theorem characterizes translations uniquely.

The homogeneous Lorentz transformations are not similarly unique. In fact, let $L$ be the conformal subgroup and $t$ any finite supertranslation. Then the group $M=tLt^{-1}$ is a subgroup of the GBM group distinct from $L$ and isomorphic to the homogeneous orthochronous Lorentz group. If $t$ is the infinitesimal supertranslation $\alpha(\theta,\phi)(\partial/\partial u)$ then the infinitesimal

elements of $M$ have the form

$$\begin{aligned} L_z' &= L_z + (L_z\alpha)(\partial/\partial u), \\ L'^{+} &= L^{+} + (L^{+}\alpha)(\partial/\partial u), \\ R_z' &= R_z + [(R_z\alpha) - \alpha\cos\theta](\partial/\partial u), \\ R'^{+} &= R^{+} + [(R^{+}\alpha) + \alpha\sin\theta e^{i\phi}](\partial/\partial u). \end{aligned} \tag{IV.24}$$

### D. Representations of the GBM Group

One powerful way to examine group structures is to look for group representations, and this method is particularly important for physics. In this subsection two theorems on representations of the GBM Lie algebra are proved.

*Theorem* IV.3. *There is at least one irreducible Hermitian representation of the* GBM *Lie algebra*; *the induced representation of the orthochronous Lorentz group is equivalent to the rest-mass zero, spin-zero representation.*

*Proof.* Consider the (indefinite) scalar product for any two functions $f(u,\theta,\phi)$ and $g(u,\theta,\phi)$:

$$\langle f,g\rangle = i\int_{-\infty}^{\infty} du \int_0^{\pi} d\theta \int_0^{2\pi} \sin\theta d\phi (\partial f/\partial u)^* g. \tag{IV.25}$$

Consider the set of all twice differentiable functions $\{f\}$ which are, together with their first two derivatives, integrable in the sense that Eq. (IV.25) remains finite when any pair of functions are integrated. Suppose that they and their first derivatives vanish at $u=\pm\infty$. With the scalar product (IV.25) one obtains a kind of Hilbert space (the scalar product is the analog of the charge density operator for a rest-mass zero boson and, therefore, is not positive definite). Consider now the linear operators

$$\begin{aligned} P_{lm}' &= P_{lm}, & R_z' &= R_z + \cos\theta, \\ L_z' &= L_z, & R'^{\pm} &= R^{\pm} + e^{\pm i\phi}\sin\theta. \\ L'^{\pm} &= L^{\pm}, \end{aligned} \tag{IV.26}$$

By direct calculation one verifies that these linear operators again obey the commutation relations of the GBM Lie algebra. Moreover, let $L'^{ab}$ be the linear operators that correspond to $L_z'$, $L'^{\pm}$, $R_z'$, and $R'^{\pm}$ via Eq. (IV.18). For the scalar product given one finds

$$\langle f, iP_{lm}g\rangle^* = \langle g, iP_{lm}f\rangle, \quad \langle f, iL^{ab}g\rangle^* = \langle g, iL^{ab}f\rangle;$$

the last relations verify the existence of a Hermitian representation.

Let us leave the question of irreducibility aside for the moment and examine the relation to representations of the Lorentz group. Consider the solutions of D'Alembert's Eq. (II.1) which are nonsingular, vanish at spatial infinity, and obey the Sommerfeld outgoing radiation condition for fixed $u$ and large $r$. Consider the quantity

$$\rho(u,\theta,\phi) = \lim[r\mu(u,r,\theta,\phi)]. \tag{IV.27}$$

Because of the outgoing radiation condition this limit always exists. Moreover, it obeys the conditions placed on $f$ and $g$ above. We shall show next that the relation between $\rho$ and $\mu$ is one-to-one.

In fact, expand the field $\mu$ in spherical harmonics $Y_{lm}$, spherical Bessel functions $j_l(kr)$ and a Fourier time integral[22]:

$$\mu(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \int_0^{\infty} k^{1/2} dk\, Y_{lm}(\theta,\phi) \times j_l(kr) e^{ikt} A_{lm}(k) + \text{c.c.} \tag{IV.28}$$

Here the $A_{lm}(k)$ are the expansion coefficients. From the Fourier-Bessel theorem[23]

$$\int_0^{\infty} r^2 dr\, j_l(kr) j_l(k'r) = \pi\delta(k-k')(2kk')^{-1}, \tag{IV.29}$$

one finds

$$A_{lm}(k) = (i\pi)^{-1} \int d^3x\, Y^*_{lm} j_l(k)^{(1/2)} \times (\partial\mu/\partial t + ik\mu)_{t=0}. \tag{IV.30}$$

Inserting the asymptotic values[23]

$$j_l(y) \underset{r\to\infty}{\longrightarrow} y^{-1}\sin(y - l\pi/2) \tag{IV.31}$$

into Eq. (IV.28) one finds

$$\rho(u,\theta,\phi) = \left\{ \sum_{l=0}^{\infty} (-i)^{l+1} \sum_{m=-l}^{l} \int_0^{\infty} k^{(1/2)} dk \times Y_{lm} e^{iku} A_{lm} \right\} + \text{c.c.} \tag{IV.32}$$

from Eq. (IV.32) one infers that the knowledge of $\rho$ enables one to calculate all the coefficients $A_{lm}$; from these coefficients one in turn obtains $\mu$. Thus, $\rho$ and $\mu$ determine each other uniquely.

If one now writes down the standard rest-mass-zero, spin-zero, representation of the Lorentz group Lie algebra acting on $\mu$, for example $P_0\mu = i(\partial\mu/\partial t)$, etc., one is able to induce a corresponding representation of differential operators acting on $\rho$. The latter turns out to be just given by the relevant quantities in Eq. (IV.26). Therefore, we have verified the equivalence of the two representations as far as the orthochronous inhomogeneous Lorentz group algebra is concerned.

The irreducibility of the given representation of the GBM Lie algebra now also follows. In fact, suppose the above GBM Lie algebra representation contained an invariant subspace. By means of the above one-to-one correspondence betwen $\rho$ and $\mu$ there would be an

[22] We carry out the proof for real $\rho$ and $\mu$. The extension to the complex case is trivial.

[23] I. N. Sneddon, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1955), Vol. 2, p. 299.

induced invariant subspace of the corresponding representation of the inhomogeneous Lorentz group algebra by operators acting on $\mu$. But the latter representation is well known to be irreducible; consequently, there is also no invariant subspace for the given GBM Lie algebra representation. This completes the proof of the theorem, Q.E.D.

The next theorem concerns an operator that commutes with all the operators of the GBM Lie algebra. Consider the "rest mass" operator

$$m^2=P_{00}{}^2-P_{10}{}^2-P_{11}P^*{}_{11}. \qquad \text{(IV.33)}$$

Using the differential operators (IV.19) one finds

$$m^2=0. \qquad \text{(IV.34)}$$

Since it is only the Lie commutators that can be taken over from the Lie algebra to representations of the Lie algebra, Eq. (IV.33) does not imply the (possibly correct) statement that the matrix corresponding to $m^2$ must vanish in every representation of the GBM Lie algebra. However, Eq. (IV.19) does imply *Theorem* IV.4. *in every representation of the* GBM *Lie algebra, $m^2$ commutes with all the operators.*

Presumably theorem IV.4 implies (though no rigorous proof is known to the author) that in any irreducible representation of the GBM group $m^2$ is represented by a constant—a linear operator which has only one eigenvalue and has every function as eigenfunction.

In the case of the Lorentz group there is a second operator, the "spin" operator, which also commutes with all the other operators.[13,16] By direct calculation one verifies that the analogous operator in the case of the GBM group fails to commute with those supertranslations that are not merely translations. Therefore, it is tempting to conjecture that the general unitary irreducible representation of the GBM group contains some mixture of Lorentz group representations with different spins. Perhaps one such representation may contain precisely the mixture of spins that is found in nature.

## V. INTEGRAL INVARIANTS OF AF SPACES

The purpose of this section is to exhibit certain integrals that can be constructed in any AF space and have very simple transformation properties under GBM transformations. The integrals were originally obtained by the quantum arguments of subsection A; the derivation given in subsection A is purely heuristic and the mathematically minded reader should simply skim through the equations without attempting to follow the arguments in detail.

### A. Quantum Considerations

Since the relation (IV.27) between the Lorentz covariant D'Alembert field $\mu$ and its Dirichlet boundary value $\rho$ at spatial infinity is one-to-one, the free rest mass zero scalar field can be quantized simply by imposing commutation relations on $\rho$. In fact, *the field $\mu$ obeys its usual commutation relations if and only if $\rho$ has the commutation relations*

$$\begin{aligned}&[\rho(u,\theta,\phi),\rho(u',\theta',\phi')]=0,\\&[\rho(u,\theta,\phi),\rho^*(u',\theta',\phi')]=-2i\hbar S(u-u')\delta(\Omega,\Omega').\end{aligned} \qquad \text{(V.1)}$$

Here square brackets denote commutators, "*" denotes adjoint, $S$ denotes the step function, and $\delta(\Omega,\Omega')$ is the invariant delta function on the surface of the two-dimensional unit sphere, e.g., $\delta(\Omega,\Omega')=(\sin\theta)^{-1}\delta(\theta-\theta')\times\delta(\phi-\phi')$. Equation (V.1) can be deduced either from the orthonormal expansions used in proving theorem IV.3 or by subjecting the standard commutator for $\mu$ to the limiting process (IV.27).

Now in any AF gravitational field there exists a quantity very similar to $\rho$, namely, the complex function $c(u,\theta,\phi)$ which appears in Eq. (II.21). This function has been discussed in great detail in the references,[3,5] where the real and imaginary parts of $\partial c/\partial u$ are called the "news functions." Here we shall need only the following three properties of $c$: (i) In order to specify a solution of the classical field equations one specifies arbitrarily the values of the news functions (and of certain other functions); (ii) the quantity $r^{-1}\partial^2c/\partial u^2$ is the amplitude of the outgoing asymptotically plane gravitational waves at infinity; (iii) the transformation law for $c$ under a GBM transformation (II.23) is

$$\begin{aligned}\bar c(\bar u,\bar\theta,\bar\phi)=Ke^{-i\eta}\{&c(u,\theta,\phi)\\&-\tfrac12\sin\theta\,\Delta[(\sin\theta)^{-1}\Delta\alpha(\theta,\phi)]\}.\end{aligned} \qquad \text{(V.2)}$$

Here $K$ and $\alpha$ are the quantities introduced in Eqs. (II.7) and (II.23), $\eta$ is twice the angle between the old and new $\theta$ directions on the sphere at infinity (e.g. on a two-space $u=$const, $r=\infty$), and $\Delta$ is defined by the equation

$$\Delta=\partial/\partial\theta+i(\sin\theta)^{-1}\partial/\partial\phi. \qquad \text{(V.3)}$$

As in the references we shall assume $(\partial c/\partial u)_{u=\pm\infty}=0$, corresponding to a Sommerfeld-type outgoing radiation condition. Note in passing that the quantity

$$\gamma\equiv c(u,\theta,\phi)-\tfrac12c(-\infty,\theta,\phi)-\tfrac12c(\infty,\theta,\phi) \qquad \text{(V.4)}$$

and the new functions have the simple transformation laws

$$\bar\gamma=Ke^{-i\eta}\gamma,\quad \partial\bar c/\partial\bar u=K^2e^{-i\eta}\partial c/\partial u. \qquad \text{(V.5)}$$

Because of properties (i) and (ii) above we can regard $c$ as the boundary value that replaces the canonical variables for the transverse modes of the gravitational field in an asymptotic treatment and attempt to quantize the field by imposing commutators on $c$. From Eq. (V.1) and dimensional arguments one can guess at possible commutation rules. In fact, *the commutation rules*

$$[c,c']=0,\quad [c,c^{*\prime}]=-2i\hbar S(u-u')\delta(\Omega,\Omega') \qquad \text{(V.6)}$$

*are covariant under arbitrary c-number* GBM *transformations. Proof.* Substitute Eqs. (II.23) and (V.2) into Eq. (V.6); then each side of the equation merely takes on the factor $K^2$, Q.E.D.

Assuming, for the sake of argument, that the commutation rules (V.6) are actually appropriate one can now ask for the generators of infinitesimal GBM transformations. In other words, let $P$ be an infinitesimal GBM transformation and suppose the functional change in $c$ under this transformation is $\bar{\delta}_P c$; we ask for a functional $I\{P\}$ which has the property

$$\bar{\delta}_P c = -[I\{P\},c]i\hbar \tag{V.7}$$

when the transformation law (V.2) and commutators (V.6) are used.

One can use the known relations of such generators in the Lorentz covariant theory to integrals over the fields to guess at a solution of Eq. (V.7). In fact, since the relation between the D'Alembert field $\mu$ and its boundary value $\rho$ is one-to-one, the energy of the field $\mu$ must be some functional of $\rho$; an explicit calculation gives for the energy $E$[24]:

$$E=\tfrac{1}{2}\int \dot{\rho}\dot{\rho}^*, \tag{V.8}$$

where

$$\int f \equiv \int_{-\infty}^{\infty} du \int_0^{\pi} d\theta \int_0^{2\pi} \sin\theta d\phi\, f(u,\theta,\phi),\quad \dot{\rho}\equiv \partial\rho/\partial u. \tag{V.9}$$

Similar expressions are obtained for the momentum and angular momentum and the Lorentz covariance of the relations can easily be verified. One can transfer the results to the gravitational case by using the spin-two nature of the gravitational field and by using dimensional arguments. In fact, *the integrals*

$$I\{\alpha(\theta,\phi)(\partial/\partial u)\} = \tfrac{1}{2}\int [\alpha\dot{c}\dot{c}^* + \dot{c}^* D\alpha + \dot{c}(D\alpha)^*], \quad \text{(a)}$$

$$\left.\begin{aligned}
&I\{L_z\} = \tfrac{1}{2}\int \dot{c}L_z\gamma^*,\\
&I\{L^{\pm}\} = \tfrac{1}{2}\int [\dot{c}L^{\pm}\gamma^* \mp 2e^{i\phi}(\sin\theta)^{-1}\dot{c}\gamma^*],\\
&I\{R_z\} = \tfrac{1}{4}\int [\dot{c}R_z\gamma^* + \dot{c}^* R_z\gamma],\\
&I\{R^{\pm}\} = \tfrac{1}{4}\int [\dot{c}R^{\pm}\gamma^* + \dot{c}^* R^{\pm}\gamma \pm 4\cot\theta e^{\pm i\phi}\dot{c}\gamma^*],
\end{aligned}\right\}\text{(b)} \tag{V.11}$$

*are solutions of Eq.* (V.7). Here

$$D\alpha \equiv -\tfrac{1}{2}\sin\theta\,\Delta[(\sin\theta)^{-1}\Delta\alpha]. \tag{V.11}$$

[24] The zero-point energy is ignored for simplicity of notation; in all the following integrals the zero-point contribution can be eliminated by a trivial factor reordering.

*Proof.* Substitute Eqs. (II.23), (IV.17), and (IV.18) into Eq. (V.10); using Eqs. (V.2), (V.4), and (V.5) one finds by partial integration that Eq. (V.7) holds in each case.

## B. Transformation Properties of the Integrals

Henceforth, the integrals (V.10) will be regarded as classical quantities that have somehow been pulled out of a hat; their transformation properties will be investigated.

Since the four infinitesimal rigid translations are uniquely defined up to a homogeneous Lorentz transformation (theorem IV.2) the corresponding integrals (V.10) are also well defined. Write these four integrals as $P_a$ ($a=0,\cdots 3$), e.g.,

$$\begin{aligned}
P_0 &= I\{\partial/\partial u\} = \tfrac{1}{2}\int \dot{c}\dot{c}^*,\\
P_1 &= I\{\sin\theta\cos\phi(\partial/\partial u)\} = \tfrac{1}{2}\int \dot{c}\dot{c}^*\sin\theta\cos\phi,\\
P_2 &= I\{\sin\theta\sin\phi(\partial/\partial u)\} = \tfrac{1}{2}\int \dot{c}\dot{c}^*\sin\theta\sin\phi,\\
P_3 &= I\{\cos\theta(\partial/\partial u)\} = \tfrac{1}{2}\int \dot{c}\dot{c}^*\cos\theta.
\end{aligned} \tag{V.12}$$

*Lemma. The integrals $P_a$ transform as a free Lorentz vector under all* GBM *transformations*; *$P_a$ is timelike or zero*; *it vanishes if and only if the news functions vanish for all retarded times u. Proof.* The transformation properties of $P_a$ are obtained by a routine calculation using Eqs. (II.23), (V.5), and (V.12). To see that $P_a$ is timelike, one can use the Schwartz inequality:

$$\left(\int ff^*\right)\left(\int gg^*\right) \geq \left(\int fg^*\right)\left(\int f^*g\right), \tag{V.13}$$

where the equality holds if and only if $f$ and $g$ are proportional. Inserting into Eq. (V.13) the pairs of functions

$$\begin{aligned}
f&=\dot{c}, & g&=\dot{c}\cos\theta,\\
f&=\dot{c}, & g&=\dot{c}\sin\theta\cos\phi,\\
f&=\dot{c}, & g&=\dot{c}\sin\theta\sin\phi,
\end{aligned} \tag{V.14}$$

using Eq. (V.12), and summing gives

$$P_aP_b\eta^{ab}\leq 0,\quad P_aP_b\eta^{ab}=0 \Rightarrow P_a=0, \tag{V.15}$$

where $\eta^{ab}$ is the Lorentz metric in the usual coordinates, Q.E.D.

As discussed in the references[3,5] $P_a$ is the total energy momentum radiated to infinity in the form of gravitational waves during the entire history of the field, the results of the references can be used to prove the

following corollary: *$P_a$ vanishes if and only if the Riemann tensor falls off as $r^{-3}$ (or faster) for all retarded times $u$.* We omit the proof.

Suppose $P_a \neq 0$. According to the lemma just proved one can then, by a GBM transformation, go into one of those frames for which

$$P_0 \neq 0, \quad P_a = 0 \quad (a \neq 0) \tag{V.16}$$

and this convention will be made.

*Theorem* V.1. *The integrals* (V.10a) *are absolute integral invariants of* AF *spaces up to a common rotation of Euclidean three-dimensional space. Proof.* By direct calculation one can verify that all supertranslations leave the integrals (V.10a) invariant. One is thus left with the conformal transformations. A conformal transformation leaves the convention (V.16) invariant if and only if it is a spatial transformation, provided $P_a \neq 0$ as we shall assume for the time being. Under a spatial rotation the $2n+1$ quantities $I\{Y_{nm}\partial/\partial u\}$, with fixed $n$ and $m=-n, -n+1, \cdots, n$, transform as spherical harmonics—e.g., as a symmetric, trace free, $n$-index tensor of Euclidean three-dimensional space. Thus the only arbitrariness in the quantities (V.10a) is that due to a transformation of the three-dimensional rotation group, which must be applied to all simultaneously. If $P_a=0$ then all the integrals (V.10) are zero and the theorem is trivially valid, Q.E.D.

Corresponding to the arbitrariness (IV.24), the transformation properties of the integrals (V.10b) are more complicated. Consider, for example, a rigid translation in the $(x,y)$ plane:

$$\bar{\theta}=\theta, \quad \bar{\phi}=\phi, \quad \bar{u}=u+\epsilon_1 \sin\theta \cos\phi + \epsilon_2 \sin\theta \sin\phi. \tag{V.17}$$

Then

$$\bar{I}\{L_z\} = I\{L_z\} + \epsilon_1 P_2 - \epsilon_2 P_1, \tag{V.18}$$

as one would expect from the fact that $I\{L_z\}$ is obviously something like the total angular momentum in the $z$ direction that is radiated to infinity by the field. Expressions similar to (V.18) hold for the transformation law of $I\{L^{ab}\}$ under any supertranslation.

The following corollary of theorem V.1 is obvious: *A necessary (but not sufficient) condition for two* AF *spaces to be isometric is that the sets of integrals* (V.10a) *for the two spaces can be brought into each other by a single rotation.* Thus, the integral invariants are geometrically interesting quantities quite apart from their rather dubious derivation as generators of infinitesimal GBM transformations in the quantized theory.

Suppose one has an AF field that is everywhere nonsingular, has no sources anywhere, and contains no stable bound state of the gravitational field so that the field consists merely of a burst of radiation that comes in, scatters itself, and then escapes back out to infinity. It then seems sensible to consider $P_a$, defined above for the outgoing waves, as the total energy of the field. What is not known is whether the incoming energy, defined in essentially the same way using an advanced time rather than a retarded time, is equal to $P_a$.[25]

## VI. CONCLUSION

An attempt was made to reconcile the crucial importance of the inhomogeneous Lorentz group in microphysics with the fact that there are dynamically very weak gravitational fields whose mere existence prevents one from giving a clean definition of the inhomogeneous Lorentz group as an isometry group. The method used was to try to interpret the Lorentz group as an asymptotic symmetry group in asymptotically flat gravitational fields.

It was found that if one uses a set of rather carefully chosen and reasonably plausible boundary conditions for the gravitational fields the asymptotic symmetry group is not the Lorentz group but the generalized Bondi-Metzner group.

It was then argued that this apparent contradiction may not be fatal because the GBM group has certain very attractive properties. First, and foremost, its structure does not depend on the particular asymptotically flat gravitational field under consideration. Second, it contains the inhomogeneous orthochronous Lorentz group as a subgroup. Third, as in the case of the inhomogeneous orthochronous Lorentz group, one has an Abelian normal subgroup whose factor group is the orthochronous homogeneous Lorentz group. Fourth, the four rigid space-time translations can be defined uniquely by the property that they form the only four-dimensional normal subgroup of the GBM group; therefore, one can hope for a Hamiltonian formalism and energy-momentum conservation laws. Fifth, at least one representation of the GBM Lie algebra essentially coincides with an irreducible Hermitian representation of the Lorentz group Lie algebra. Finally, it is possible to introduce a rest-mass operator which commutes with all the elements of the GBM Lie algebra. One nontrivial difference was noted: It seems to be impossible to introduce a spin operator that commutes with all the elements of the GBM Lie algebra.

Among the many unsolved questions connected with the work discussed here three seem to the author to be particularly interesting. First, does the GBM group permit representations for which the rest-mass operator differs from zero; do these representations, if they exist, correspond physically to a mixture of different spins? Second, what can one say about time-reversal transformations within the kind of approach used here? Third, can one use the fact that the group structure is metric independent to create an $S$-matrix theory of gravitational waves? The second and third questions

[25] The author is indebted to P. G. Bergmann, J. N. Goldberg, J. Knight, C. Misner, and R. Schiller for discussions of the material in Sec. V and to Dr. Goldberg for making available the contents of his paper "Asymptotic Invariants" prior to publication.

are being actively investigated. After repeated failures the author has given up the first question as being beyond his own mathematical capabilities, but he hopes this paper will stimulate a competent investigation of the question.

## ACKNOWLEDGMENTS

J. Ehlers of Hamburg and Syracuse Universities suggested many of the main ideas of this paper; several conversations with A. Trautman, Warsaw University, and F. A. E. Pirani, King's College, London, during the earlier stages of the investigation were very helpful. The continued advice and encouragement of H. Bondi, and his kindness in making his results available prior to publication, were of major assistance. The calculations for theorem IV.3 were carried through jointly with J. Knight, USASRDL. If a reasonable degree of clarity and rigor has been achieved then this result is due mainly to sharp and very useful criticisms by A. Komar and P. G. Bergmann, Syracuse University, on previous versions of the theorems. Section III is almost entirely due to a conversation with J. N. Goldberg, ARL, and to the kindness of E. Newman and T. Unti, Pittsburgh University, in making their results available prior to publication. Several conversations with R. Arnowitt, Northeastern University, S. Deser, Brandeis, and especially C. W. Misner, Princeton, were important because they indicated that the results given here are not necessarily incompatible with the results obtained in the Arnowitt-Deser-Misner papers.

This work was performed under the direction of Dr. W. S. McAfee, USASRDL, whose personal kindness and keenness in discussing ideas made the job a pleasant one. The encouragement of Dr. H. J. Merrill, USASRDL, and his constant willingness to help in administrative matters made the work reported on here possible.

PHYSICAL REVIEW VOLUME 128, NUMBER 6 DECEMBER 15, 1962

# Incoherence, Quantum Fluctuations, and Noise

I. R. Senitzky
*U. S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey*
(Received June 6, 1962)

An examination is made of the relationship between the uncertainty principle and minimum amplifier noise. First, the concept of coherence is discussed, and an incoherence parameter is defined in terms of the uncertainty that enters into the uncertainty principle. Harmonic oscillator states are examined for coherence. The concept of noise is then discussed and contrasted with incoherence, noise referring to behavior in time of a single system while incoherence involving comparison among members of an ensemble. It is shown, with illustrations, that the two concepts are different, and that an incoherent field of a cavity mode need not exhibit noise. In particular, the zero-point field in a lossless cavity is not noise. The superposition of many incoherent effects, however, usually leads to noise. Spontaneous emission is examined both for coherence and noise. It is shown that the spontaneous emission field of a single molecule is incoherent but does not exhibit noise; the (low order) spontaneous emission from a molecular beam, however, does constitute noise. Spontaneous emission from complex systems is also discussed. The origin of fundamental noise in an amplifier is investigated and is shown to come from spontaneous emission by the amplification mechanism. It is concluded that fundamental noise cannot be determined by a consideration of quantum fluctuations of—or by the application of the uncertainty principle to—the electromagnetic field only, as has been done in several recent articles. The physical significance of the zero-point field is analyzed, and is shown to lie in a formal contribution to spontaneous emission by the mechanism coupled to the field, provided this mechanism is treated quantum mechanically.

## INTRODUCTION

In recent years, there has arisen an interest in "fundamental" noise, noise which has been attributed to fundamental physical laws or phenomena, such as the uncertainty principle, quantum fluctuations, or spontaneous emission, and which cannot be eliminated in principle. This interest is due, in large part, to the development of maser amplifiers, in which the noise is so low as to offer the possibility of approaching, indeed, the level of fundamental noise.

There has been discussion[1,2] of fundamental noise that is based mainly on the uncertainty principle or on quantum fluctuation, and, as will be shown, is more or less unsatisfactory. A related unsatisfactory situation exists with respect to the concept of coherence (which is associated with both noise and the uncertainty principle), because of the various different meanings attached to the word "coherent." It is the purpose of the present article to discuss the concept of coherence, and then to analyze the relationship between noise on the one hand, and quantum fluctuations and the uncertainty principle on the other.

Coherence is discussed in Part I, and noise is the chief topic of Part II. Various aspects of spontaneous emission are considered in Part III, and the source of fundamental amplifier noise is discussed in Part IV.

[1] R. Serber and C. H. Townes, in *Quantum Electronics*, edited by C. H. Townes (Columbia University Press, New York, 1960).
[2] W. H. Louisell, A. Yariv, and A. E. Siegman, Phys. Rev. **124**, 1646 (1961).