significance. In the static model, a reasonably accurate physical eigenstate has been constructed,[10] and used to compute the moments. The result, for a reasonable value of the coupling constant, was a vector moment in good agreement with experiment, but a scalar moment which was much too large. Other static model computations have given similar results.

It seems fair to ask whether such results reflect a real failure of pseudoscalar meson theory, or whether the rather drastic assumptions of the static model might be responsible for the lack of agreement. Our results with the Lee model argue mildly in favor of the latter interpretation. Of course, we would not be so rash as to claim that our numerical results have any relevance to the problem of the nucleon moments. However, if the neglect of recoil and of pairs alter the magnetic moment prediction in this model theory, it seems at least plausible that they may likewise do so in a more realistic one.

ACKNOWLEDGMENTS

The authors are grateful to Professor F. Fender and the staff of the Rutgers Computation Laboratory for assistance in carrying out the numerical computations. We also wish to thank Professor R. Blankenbecler for several useful discussions.

PHYSICAL REVIEW VOLUME 128, NUMBER 6 DECEMBER 15, 1962

Perturbation Theory of Many-Boson Systems*

A. J. KROMMINGA† AND M. BOLSTERLI
School of Physics, University of Minnesota, Minneapolis, Minnesota
(Received June 7, 1962)

A noncanonical transformation of the boson creation and annihilation operators is performed in order to obtain a Hamiltonian which can be treated by the standard methods of field-theoretic perturbation theory. The standard results of Belyaev (with a slight modification) are rederived by this technique.

I. INTRODUCTION

It has been shown by Belyaev[1] that the many-boson system can be treated by the methods of field-theoretic perturbation theory. His proof of this fact, however, is outside the realm of field-theoretic perturbation theory. The purpose of this paper is an alternative derivation of Belyaev's result, using only standard perturbative techniques. This new derivation is somewhat more exact in its treatment of the zero-momentum state, and a slight correction to Belyaev's result is found. Further light is thrown on the nature of the approximation of large numbers.

II. FORMULATION OF THE PROBLEM

We consider a system consisting of a large number of identical bosons interacting via two-body forces. The units are chosen so that $\hbar=2m=1$, where m is the mass of a single boson. Then the Hamiltonian for the system is

$$\mathcal{E}=\sum_{\mathbf{k}} k^2a_{\mathbf{k}}^\dagger a_{\mathbf{k}}+(2V)^{-1}\sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4} \times v(\mathbf{k}_1-\mathbf{k}_3)\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}a_{\mathbf{k}_1}{}^\dagger a_{\mathbf{k}_2}{}^\dagger a_{\mathbf{k}_3}a_{\mathbf{k}_4}. \quad (1)$$

We use script type for operators, except the $a_{\mathbf{k}}$, $a_{\mathbf{k}}{}^\dagger$, where the $a_{\mathbf{k}}$, $a_{\mathbf{k}}{}^\dagger$, are the annihilation and creation operators, respectively, for a boson in the single-particle state with wave number $\mathbf{k}$ and wave function $V^{-1/2}e^{i\mathbf{k}\cdot\mathbf{r}}$. V is the volume of the system and only those $\mathbf{k}$'s necessary for completeness of the set of single-particle functions are included, i.e., the $\mathbf{k}$'s satisfying the usual periodic boundary conditions. $v(\mathbf{k})$ is the Fourier transform of the two-body interaction $u(\mathbf{r})$:

$$v(\mathbf{k})=\int u(\mathbf{r})e^{i\mathbf{k}\cdot\mathbf{r}}d^3r. \quad (2)$$

In order that $\mathcal{E}$ be Hermitian, we must have

$$v(-\mathbf{k})=v^*(\mathbf{k}). \quad (3)$$

Since the $a_{\mathbf{k}}$, $a_{\mathbf{k}}{}^\dagger$ are Bose operators, they obey the commutation relations

$$[a_{\mathbf{k}}{}^\dagger,a_{\mathbf{k}'}{}^\dagger]=[a_{\mathbf{k}},a_{\mathbf{k}'}]=0, \quad [a_{\mathbf{k}},a_{\mathbf{k}'}{}^\dagger]=\delta_{\mathbf{k},\mathbf{k}'}. \quad (4)$$

From these, it follows that $\mathfrak{N}_{\mathbf{k}}=a_{\mathbf{k}}{}^\dagger a_{\mathbf{k}}$ is the operator for the number of particles in the single-particle state $\mathbf{k}$ and has the eigenvalues $0, 1, 2, \cdots$. The total number operator is

$$\mathfrak{N}=\sum_{\mathbf{k}}\mathfrak{N}_{\mathbf{k}}. \quad (5)$$

$\mathfrak{N}$ and $\mathcal{E}$ commute, so that we can now formulate the problem as follows: find the simultaneous eigenvectors, $|E_i,N\rangle$ of $\mathcal{E}$ and $\mathfrak{N}$ and, in particular, the set of eigen-

* Supported in part by the U. S. Atomic Energy Commission.
† Present address: Physics Department, Iowa State University, Ames, Iowa.
[1] S. T. Belyaev, J. Exptl. Theoret. Phys. (U.S.S.R.) 34, 417 (1958) [translation: Soviet Phys.—JETP 7, 289 (1958)].

values E_i which occurs for a fixed value of N:

$$\mathcal{E}|E_i,N\rangle = E_i|E_i,N\rangle,\qquad \mathfrak{N}|E_i,N\rangle = N|E_i,N\rangle. \tag{6}$$

Of course, we shall actually consider just the lowest eigenvalues of the energy for a fixed number of particles N; the lowest energy eigenvalue, the ground-state energy, and its eigenvector are of particular interest.

Consider first just the kinetic energy part of the Hamiltonian:

$$\mathcal{T} = \sum_{\mathbf{k}} k^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} = \sum_{\mathbf{k}} k^2 \mathfrak{N}_k. \tag{7}$$

We know the eigenvectors of $\mathcal{T}$: they are just the simultaneous eigenvectors $|N_k\rangle$ of all the $\mathfrak{N}_k$ (the $\mathfrak{N}_k$ commute with each other)

$$\mathfrak{N}_{\mathbf{j}}|N_{\mathbf{k}}\rangle = N_{\mathbf{j}}|N_{\mathbf{k}}\rangle,\qquad \mathcal{T}|N_{\mathbf{k}}\rangle = \sum_{\mathbf{j}} N_{\mathbf{j}} j^2 |N_{\mathbf{k}}\rangle. \tag{8}$$

Thus, the ground-state eigenvector for the system consisting of N noninteracting bosons ($v=0$) is just the vector

$$|N_0=N,\quad N_{\mathbf{k}}=0\rangle,\quad \mathbf{k}\neq 0, \tag{9}$$

with eigenvalue $T=E=0$.

The idea of perturbation theory is then to expand a state $|E_i,N\rangle$ of the interacting system in terms of the unperturbed states (eigenstates of the interactionless Hamiltonian):

$$|E_i,N\rangle = \sum_{\mathbf{k}} |N_{\mathbf{k}}\rangle\langle N_{\mathbf{k}}|E_i,N\rangle,\qquad \sum_{\mathbf{k}} N_{\mathbf{k}} = N \tag{10}$$

and find the coefficients $\langle N_{\mathbf{k}}|E_i,N\rangle$ and eigenvalues E_i as power series in the interaction coupling constant.

Field-theoretic techniques have greatly simplified the solution of this problem. However, the application of such techniques is not feasible unless the unperturbed ground state $|0\rangle_u$ is a "vacuum"; that is, a state such that

$$a_{\mathbf{k}}|0\rangle_u = 0 \tag{11}$$

for all $\mathbf{k}$ of a complete set. In our case, the state $|0\rangle_u$ satisfies

$$a_{\mathbf{k}}|0,N\rangle_u = \delta_{\mathbf{k},0} N^{1/2}|0,\, N-1\rangle_u \neq 0. \tag{12}$$

It is not possible, as with fermions, to just relabel annihilation and creation operators and thus avoid the difficulty, since

$$a_0^\dagger|0,N\rangle_u = (N+1)^{1/2}|0,\, N+1\rangle_u \neq 0. \tag{13}$$

In the past, the 0-momentum single-particle state has, therefore, been treated only approximately[2] (except by Belyaev[1]). The standard approximation has been the replacement of a_0 and $a_0^\dagger$ by a c number $N_0^{1/2}$ which is later determined so as to minimize the ground-state energy of the system. This eliminates the 0-momentum state and makes $|0\rangle_u$ a vacuum for the operators which remain, but it has the disadvantage that terms of the type $N_0^{1/2}\,(2V)^{-1}a_1^\dagger a_2 a_3$ arise in $\mathcal{E}$. These terms do not commute with $\mathfrak{N}$, so that it is no longer possible to find energy eigenvalues for a system with a fixed number of particles.

We, therefore, introduce a new transformation to eliminate the 0-momentum state. The new transformation has the advantages that (a) it involves no approximation, as does the replacement of a_0 and $a_0^\dagger$ by $N_0^{1/2}$, and (b) it, therefore, introduces no number nonconserving terms into the Hamiltonian.

III. TRANSFORMATION TO EXCITATION OPERATORS

From the commutation relations (4), we have

$$a_0\mathfrak{N}_0 = (\mathfrak{N}_0+1)a_0. \tag{14}$$

and, therefore, by the spectral theorem,

$$a_0 f(\mathfrak{N}_0) = f(\mathfrak{N}_0+1)a_0. \tag{15}$$

Consider, then, the operators

$$b_{\mathbf{k}} = \mathfrak{N}_0^{-1/2}a_0^\dagger a_{\mathbf{k}} = a_0^\dagger(\mathfrak{N}_0+1)^{-1/2}a_{\mathbf{k}},\qquad b_{\mathbf{k}}^\dagger = a_{\mathbf{k}}^\dagger a_0\mathfrak{N}_0^{-1/2} = a_{\mathbf{k}}^\dagger(\mathfrak{N}_0+1)^{-1/2}a_0. \tag{16a}$$

Clearly,

$$b_0 = b_0^\dagger = \mathfrak{N}_0^{1/2}. \tag{16b}$$

Also, since $a_0a_0^\dagger = \mathfrak{N}_0+1$, we have

$$b_{\mathbf{k}}^\dagger b_{\mathbf{k}'} = a_{\mathbf{k}}^\dagger a_{\mathbf{k}'},\qquad b_{\mathbf{k}'}b_{\mathbf{k}}^\dagger = a_{\mathbf{k}'}a_{\mathbf{k}}^\dagger,\quad \mathbf{k}\neq 0,\quad \mathbf{k}'\neq 0. \tag{17}$$

Therefore, for $\mathbf{k}\neq 0$, $\mathbf{k}'\neq 0$, we have

$$[b_{\mathbf{k}}^\dagger, b_{\mathbf{k}'}^\dagger] = [b_{\mathbf{k}}, b_{\mathbf{k}'}] = 0,\quad [b_{\mathbf{k}}, b_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k},\mathbf{k}'}, \tag{18}$$

and the $b_{\mathbf{k}}$, $b_{\mathbf{k}}^\dagger$ are annihilation and creation operators for quasi-particle excitations of the system which obey Bose statistics and which are just like particles in the sense that their annihilation and creation operators satisfy the same commutation relations that Bose particle annihilation and creation operators satisfy.

Equation (17) apparently does not hold if both sides are applied to a state with $N_0=0$. However, we will simply specify that products of operators are to be simplified *before* being applied to state vectors. This avoids the difficulty and, moreover, just corresponds to the techniques used in field-theoretic perturbation theory, i.e., one deals for the most part directly with the operators alone. Another way of seeing that Eq. (17) is correct is to note that $\mathcal{E}$, Eq. (25), in terms of b's with the commutation relations (18) acts on any state in exactly the same manner as $\mathcal{E}$, Eq. (1), in terms of the a's; both forms of $\mathcal{E}$, therefore, have the same spectrum.

The a's do not commute with $\mathfrak{N}$, since they change the number of particles; however, the b's change the number of quasi-particles but leave the total number of

[2] N. N. Bogolyubov, J. Phys. (U.S.S.R.) **11**, 23 (1947); K. A. Brueckner and K. Sawada, Phys. Rev. **106**, 1117 (1957); T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

particles unchanged and, therefore, commute with $\mathfrak{N}$ (as can easily be checked). Thus, there are no longer any particle number nonconserving operators present. If we start with a state with N particles, then all the intermediate states which arise after a certain number of b's have been applied have the same number, N, of particles. That is, we can completely eliminate the 0-momentum state by first replacing the a's by b's, and then giving $\mathfrak{N}_0$ in terms of the $b_{\mathbf{k}}$, $\mathbf{k}\neq 0$:

$$\mathfrak{N}_0 = N - \sum_{\mathbf{k}}{}' \mathfrak{N}_k, \tag{19}$$

where N is a c number; we have introduced the notation

$$\sum_{\mathbf{k}}{}' = \sum_{\mathbf{k}\neq 0}, \tag{20}$$

which will be used from now on. The subsidiary condition has now been taken care of and we need only translate $\mathcal{E}$ into b language.

Obviously, from (17),

$$\mathcal{T} = \sum_{\mathbf{k}}{}' k^2 b_{\mathbf{k}}^\dagger b_{\mathbf{k}} = \sum_{\mathbf{k}}{}' k^2 \mathfrak{N}_{\mathbf{k}}. \tag{21}$$

For the interaction term we first write, using (4) and dropping the $\mathbf{k}$'s for the moment in an obvious notation:

$$\begin{aligned}\mathcal{E}_{\text{int}} &= (4V)^{-1} \sum [v(1-3)+v(2-4)]\delta_{1+2,3+4} a_1^\dagger a_2^\dagger a_3 a_4 \\ &= (4V)^{-1} \sum [v(1-3)+v(3-1)]\delta_{1+2,3+4} a_1^\dagger a_2^\dagger a_3 a_4 \\ &= (2V)^{-1} \sum w(1-3)\delta_{1+2,3+4} a_1^\dagger a_2^\dagger a_3 a_4,\end{aligned} \tag{22}$$

where

$$w(\mathbf{k}) = \tfrac{1}{2}[v(\mathbf{k})+v(-\mathbf{k})], \quad w(\mathbf{k}) = w(-\mathbf{k}) = w^*(\mathbf{k}). \tag{23}$$

Now, we use (17) and (18) to write

$$\begin{aligned}\mathcal{E}_{\text{int}} &= (2V)^{-1} \sum w(1-3) \\ &\qquad \times \delta_{1+2,3+4}(a_1^\dagger a_3 a_2^\dagger a_4 - \delta_{2,3} a_1^\dagger a_4) \\ &= (2V)^{-1} \sum w(1-3) \\ &\qquad \times \delta_{1+2,3+4}(b_1^\dagger b_3 b_2^\dagger b_4 - \delta_{2,3} b_1^\dagger b_4).\end{aligned} \tag{24}$$

Going over from $\sum$ to $\sum'$ requires taking out all the zero momentum subscripts; the result is, with (16b):

$$\begin{aligned}\mathcal{E}_{\text{int}} &= E_1 + \mathcal{E}_2' + \mathcal{E}_2'' + \mathcal{E}_3 + \mathcal{E}_4 \\ E_1 &= [w(0)/2V]N(N-1), \\ \mathcal{E}_2' &= (N/V)\sum_{\mathbf{k}}{}' w(\mathbf{k})\mathfrak{N}_{\mathbf{k}} - V^{-1}\sum_{\mathbf{k}'}{}' \mathfrak{N}_{\mathbf{k}'} \sum_{\mathbf{k}}{}' w(\mathbf{k})\mathfrak{N}_{\mathbf{k}} \\ \mathcal{E}_2'' &= \sum_{\mathbf{k}}{}' [w(\mathbf{k})/2V]\{b_{\mathbf{k}}^\dagger b_{-\mathbf{k}}^\dagger [\mathfrak{N}_0(\mathfrak{N}_0-1)]^{1/2} \\ &\qquad + [\mathfrak{N}_0(\mathfrak{N}_0-1)]^{1/2} b_{\mathbf{k}} b_{-\mathbf{k}}\}, \\ \mathcal{E}_3 &= \sum_{\mathbf{k}\mathbf{k}'}{}' [w(\mathbf{k})/V](\mathfrak{N}_0^{1/2} b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} + b_{\mathbf{k}}^\dagger b_{\mathbf{k}'}^\dagger b_{\mathbf{k}+\mathbf{k}'} \mathfrak{N}_0^{1/2}) \\ \mathcal{E}_4 &= (2V)^{-1} \sum_{\mathbf{k}_1\neq\mathbf{k}_3}{}' w(1-3)\delta_{1+2,3+4} b_1^\dagger b_2^\dagger b_3 b_4.\end{aligned} \tag{25}$$

It is interesting to see just how the subsidiary condition is taken care of in (25). $\mathcal{E}_2'$ and $\mathcal{E}_4$ change neither the number of quasi-particles nor the number of particles. $\mathcal{E}_2''$ and $\mathcal{E}_3$, however, do change the number of quasiparticles; the square-root factors just keep the number of quasi-particles from exceeding the number of particles. They thus insure that the states with the total number of quasi-particles less than or equal to N form an invariant subspace under $\mathcal{E}$. If we start with a state in this subspace, as we shall, none of the terms in (25) will take us out of the subspace. In a moment we shall see that one of the approximations we make amounts to introducing terms which destroy the invariance of the subspace, but only to relative order N^{-1} or V^{-1}.

Before doing perturbation theory on (25), we put it in normal order. $\mathcal{E}_4$ is already normal-ordered. Using the fact that the normal pairings (np) are

$$\begin{aligned}&\mathrm{np}(b_{\mathbf{k}} b_{\mathbf{k}'}) = \mathrm{np}(b_{\mathbf{k}}^\dagger b_{\mathbf{k}'}^\dagger) = \mathrm{np}(b_{\mathbf{k}}^\dagger b_{\mathbf{k}'}) = 0, \\ &\mathrm{np}(b_{\mathbf{k}} b_{\mathbf{k}'}^\dagger) = \delta_{\mathbf{k},\mathbf{k}'},\end{aligned} \tag{26}$$

we obtain for $\mathcal{E}_2'$

$$\mathcal{E}_2' = [(N-1)/V]\sum_{\mathbf{k}}{}' w(\mathbf{k})\mathfrak{N}_{\mathbf{k}} - V^{-1} : \sum_{\mathbf{k}'}{}' \mathfrak{N}_{\mathbf{k}'} \sum_{\mathbf{k}}{}' w(\mathbf{k})\mathfrak{N}_{\mathbf{k}} :. \tag{27}$$

In order to put $\mathcal{E}_2''$ and $\mathcal{E}_3$ in normal order, we must expand the square roots:

$$\mathfrak{N}_0^{1/2} = (N - \sum_{\mathbf{k}}{}' \mathfrak{N}_{\mathbf{k}})^{1/2} = N^{1/2} \sum_r \binom{\frac{1}{2}}{r} (-)^r (\sum_{\mathbf{k}}{}' \mathfrak{N}_{\mathbf{k}}/N)^r. \tag{28}$$

Then

$$\begin{aligned}&(\sum_{\mathbf{k}}{}' \mathfrak{N}_{\mathbf{k}}/N)^r b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} \\ &\quad = :(\sum_{\mathbf{k}}{}' \mathfrak{N}_{\mathbf{k}}/N)^r b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'}: \\ &\quad + [r(r+1)/2N] : (\sum_{\mathbf{k}'}{}' \mathfrak{N}_{\mathbf{k}'}/N)^{r-1} b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} : + \cdots,\end{aligned} \tag{29}$$

so that

$$\begin{aligned}&\mathfrak{N}_0^{1/2} b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} \\ &\quad = N^{1/2} \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} (-)^r : (\sum_{\mathbf{k}'}{}' \mathfrak{N}_{\mathbf{k}'}/N)^r b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} : \\ &\qquad + N^{-1/2} \sum_{r=0}^{\infty} \binom{\frac{1}{2}}{r} (-)^r \\ &\qquad\quad \times (r(r+1)/2) : (\sum_{\mathbf{k}'}{}' \mathfrak{N}_{\mathbf{k}'}/N)^{r-1} b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} : + \cdots \\ &\quad = N^{1/2} \sum_{r=0}^{\infty} \left[\binom{\frac{1}{2}}{r} (-)^r + \binom{\frac{1}{2}}{r+1} (-)^{r+1} (r+1)(r+2)/2N \right] \\ &\qquad\quad \times : (\sum_{\mathbf{k}'}{}' \mathfrak{N}_{\mathbf{k}'}/N)^r b_{\mathbf{k}+\mathbf{k}'}^\dagger b_{\mathbf{k}} b_{\mathbf{k}'} :.\end{aligned} \tag{30}$$

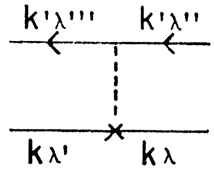


FIG. 1. The vertex from the second term of $\mathcal{E}_2'$.

We now specify that we are going to consider only systems with N and V so large that we can neglect terms of relative order N^{-1} or V^{-1} wherever they appear. Then only the first term in (30) need be retained and we see that

$$\mathcal{E}_3 = :\sum_{\mathbf{kk}'}{}' [w(\mathbf{k})/V](\mathfrak{N}_0^{1/2} b_{\mathbf{k}+\mathbf{k}'}^\dagger b_\mathbf{k} b_{\mathbf{k}'} + b_\mathbf{k}^\dagger b_{\mathbf{k}'}^\dagger b_{\mathbf{k}+\mathbf{k}'} \mathfrak{N}_0^{1/2}): \quad (31)$$

and, by a similar procedure,

$$\mathcal{E}_2'' = :\sum_\mathbf{k}{}' [w(\mathbf{k})/2V](b_\mathbf{k}^\dagger b_{-\mathbf{k}}^\dagger \mathfrak{N}_0 + \mathfrak{N}_0 b_\mathbf{k} b_{-\mathbf{k}}): \quad (32)$$

It is just here that we have destroyed the invariance of the subspace with N or fewer than N quasi-particles. As we have just seen, however, we only introduce errors of relative order N^{-1} or V^{-1}; the correction only affects that part of a state which has $N_0 \leqslant 1$. Moreover, there is no particle nonconservation, but the number of quasi-particles is allowed to be greater than the number of particles.

IV. PERTURBATION THEORY

We now treat the Hamiltonian by using field-theoretic perturbation theory. For the time being we will drop E_1, since it is just a constant. We let

$$\mathcal{E}_u = \mathcal{T} + (N/V)\sum_\mathbf{k}{}' w(k)\mathfrak{N}_\mathbf{k} = \sum_\mathbf{k}{}'[k^2+(N/V)w(\mathbf{k})]\mathfrak{N}_\mathbf{k} = \sum_\mathbf{k}{}'\epsilon(\mathbf{k})\mathfrak{N}_\mathbf{k} \quad (33)$$

be our "unperturbed" Hamiltonian, where we have used the fact that $N-1$ is the same as N, owing to the size of the system. Again, the eigenvectors of $\mathcal{E}_u$ are just the $|N_\mathbf{k}\rangle'$, where the prime means that only the $N_\mathbf{k}$ for $k\neq 0$ are given. The eigenvalues of $\mathcal{E}_u$ are then

$$\mathcal{E}_u|N_j\rangle' = \sum_\mathbf{k}{}' \epsilon(\mathbf{k})N_\mathbf{k}|N_j\rangle'. \quad (34)$$

If $\epsilon(\mathbf{k})>0$, for $\mathbf{k}\neq 0$, then the unperturbed ground state is the state $|0\rangle'$:

$$b_\mathbf{k}|0\rangle'=0, \quad \mathbf{k}\neq 0. \quad (35)$$

We treat only the case in which $\epsilon(\mathbf{k})>0$ for all $\mathbf{k}$; this means that the two-body force is repulsive, since we have

$$w(0)=\int u(\mathbf{r})d^3r>0. \quad (36)$$

We now go over directly to the diagrammatic representation of the S operator without giving the intermediate analysis, since the latter is of the standard type. The results of this straightforward analysis are the following correspondence between the matrix elements of the S operator and Feynman diagrams.

To a directed quasi-particle line, with wave number $\mathbf{k}$ and frequency λ, corresponds an unperturbed quasi-particle Green's function $G_0^u(\mathbf{k},\lambda)$:

$$G_0^u(\mathbf{k},\lambda) = (i/2\pi)[\lambda-\epsilon(\mathbf{k})+i0]^{-1}. \quad (37)$$

The second term in (27) gives the vertex shown in Fig. 1 with corresponding factor

$$[2\pi i w(\mathbf{k})/V]\delta(\lambda'''+\lambda'-\lambda''-\lambda). \quad (38)$$

The N terms in (32) give Figs. 2(a) and 2(b), each corresponding to the same factor

$$[-2\pi i N w(\mathbf{k})/V]\delta(\lambda+\lambda'), \quad (39)$$

where the factor 1/2 goes out because the term always contributes twice. The $\sum_\mathbf{k}{}' \mathfrak{N}_\mathbf{k}$ terms in (32) give Figs. 2(c) and 2(d) with the common factor

$$[2\pi i w(\mathbf{k})/V]\delta(\lambda+\lambda'+\lambda''-\lambda'''). \quad (40)$$

In $\mathcal{E}_3$ (31), the square roots must be expanded in order to obtain the correspondence with Feynman diagrams; the result is Figs. 3(a) and 3(b), each with the corresponding series

$$[-2\pi i w(\mathbf{k}_1)/V]\sum_{m=0}^{\infty}[(d/dx)^m(N-x)^{1/2}]_{x=0} \times\delta(\lambda_1+\lambda_2+\sum\lambda^{(i)}-\lambda_3-\sum\mu^{(i)}). \quad (41)$$

Finally, $\mathcal{E}_4$ (25) gives Fig. 4 and

$$[-2\pi i w(\mathbf{k}_1-\mathbf{k}_3)/V]\delta_{\mathbf{k}_1+\mathbf{k}_2,\mathbf{k}_3+\mathbf{k}_4}\delta(\lambda_1+\lambda_2-\lambda_3-\lambda_4). \quad (42)$$

In addition to these vertex and quasi-particle line factors, there are just a few combinatorial ones: first, a factor $(r!)^{-1}$ for each set of r identical pieces in a

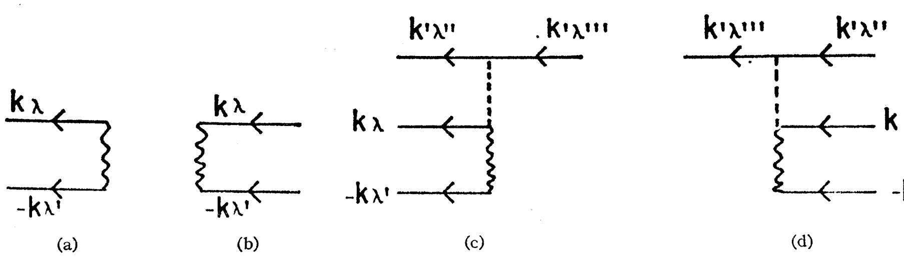


FIG. 2. Vertices from $\mathcal{E}_2''$.

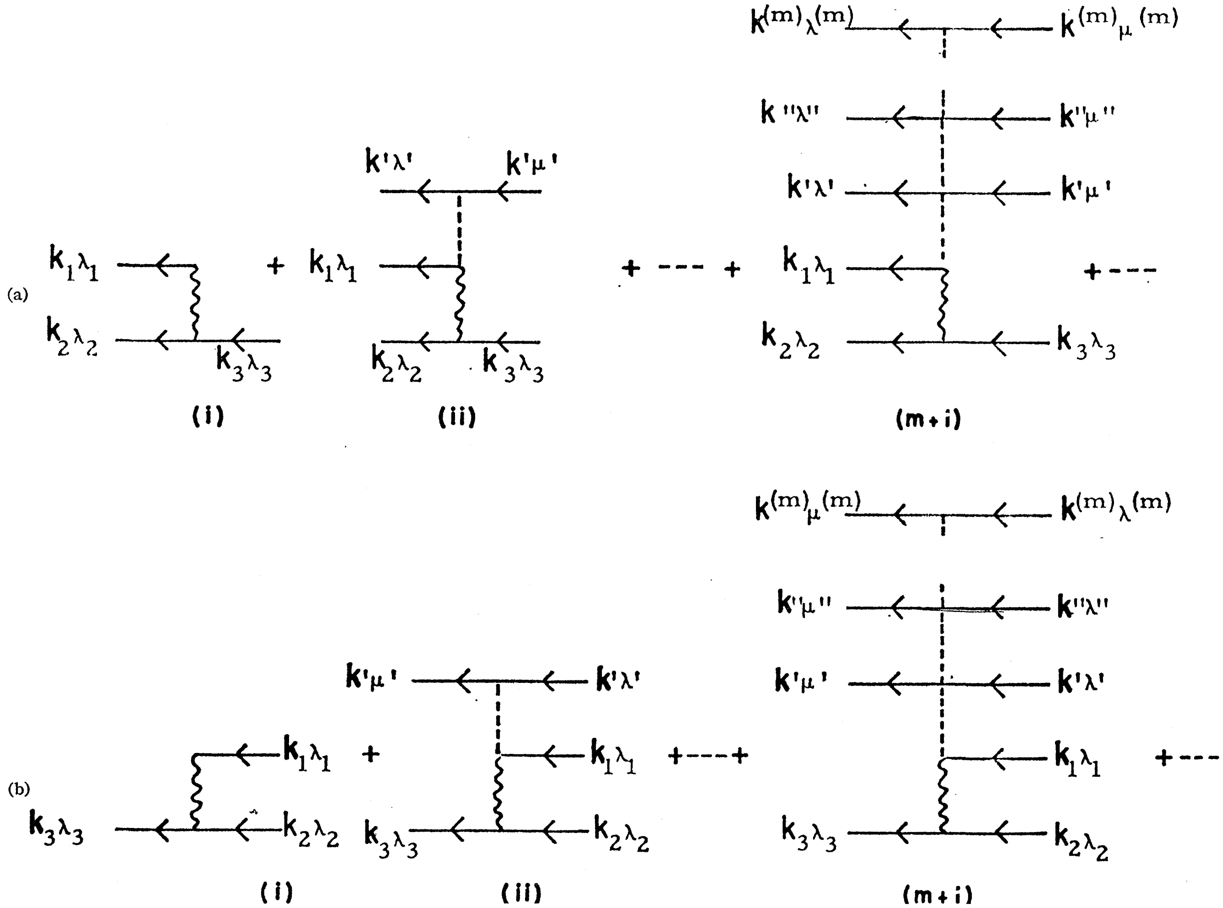


FIG. 3. Vertices from $\mathcal{E}_3$.

diagram, and second, an extra factor of 1/2 for certain isolated strongly connected graphs (definition later). This latter factor is to compensate for the fact that in certain cases one too many factors of 1/2 have been taken out. As usual, intermediate **k**'s are to be summed over their possible values except zero, and intermediate λ's are to be integrated from $-\infty$ to ∞.

We shall now show that the dashed lines shown in certain of the vertices can be eliminated by a simple procedure. We shall then be left with the vertices without dashed lines, Figs. 2(a), 2(b), 3(a)(*i*), 3(b)(*i*) and 4. These latter are just what one has in a theory which starts by replacing $a_0^\dagger$ and a_0 by $N^{1/2}$ everywhere in $\mathcal{E}$ (1). However, we shall see that the elimination of the dotted line leaves its mark. First, it changes the single quasi-particle energies, that is to say, it leads to self-energy terms. Second, the factor that remains in the vertices without dashed lines is not $N^{1/2}$, but rather a "renormalized" value which turns out to be $\langle\mathfrak{N}_0\rangle^{1/2}$.

V. REMOVAL OF DASHED LINES; THE RENORMALIZED PROBLEM

Some terminology will now be introduced. A *graph* is part of a diagram; a *connected* graph is one with no disconnected parts; a *strongly connected* graph is a connected graph which cannot be separated into two or more disconnected parts by cutting dashed lines only; and an *isolated* strongly connected graph is a strongly connected graph with no external quasi-particle lines.

Consider a dashed line within a strongly connected graph, for example, Fig. 5(a), which uses the vertex of Fig. 1. By assumption the graph obtained by removing the dashed line is also a strongly connected graph, for example, Fig. 5(b) corresponds to removing the dashed line from Fig. 5(a). Now the ratio of the factors associated with these two graphs is of order V^{-1}, since there is the same number of **k** sums in both and, therefore, the extra V^{-1} from (38) is not compensated by a V arising from a possible extra **k** sum. We may, therefore, neglect any diagram that has a dashed line entirely within one of its strongly connected graphs.

It is easy to see that to each isolated strongly con-

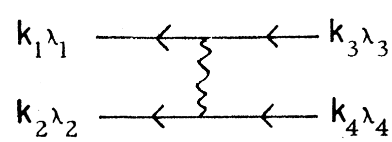


FIG. 4. The vertex from $\mathcal{E}_4$.

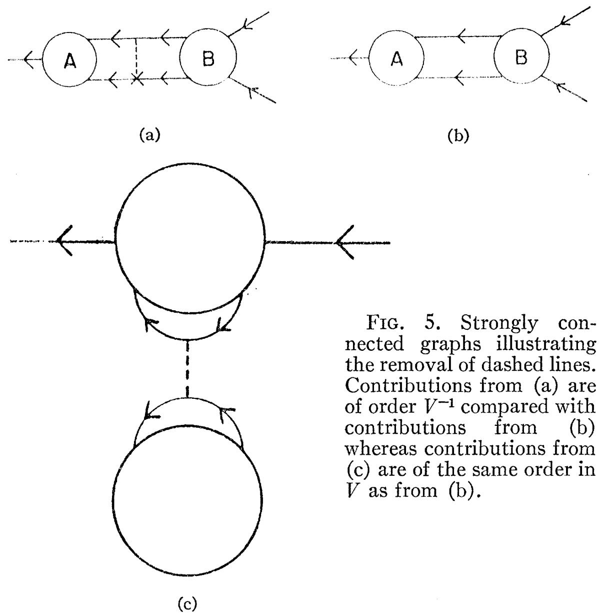


FIG. 5. Strongly connected graphs illustrating the removal of dashed lines. Contributions from (a) are of order V^{-1} compared with contributions from (b) whereas contributions from (c) are of the same order in V as from (b).

nected graph without internal dashed lines corresponds a factor of order V. This follows directly from an analysis of the number of factors N and V and the number of independent $\mathbf{k}$ sums associated with such a graph and has been noted previously.[1,2]

Consider then a strongly connected graph which is connected to the rest of the diagram by a single dashed line, for example, Fig. 5(c). If both parts which are connected by the single dashed line have external quasi-particle lines, then the argument given just above applies and the diagram is negligible. However, if one or both of the parts is isolated, then the V^{-1} from the dashed line is compensated by a factor V due to the isolated strongly connected graph. Clearly, a diagram in which two strongly connected parts, at least one of which is isolated, are connected by s dashed lines is of relative order V^{-s+1} and, therefore, negligible for $s>1$.

We thus have the following general result: The dashed lines need only be considered when they connect strongly connected graphs. To be more precise, a connected graph with dashed lines is non-negligible if, and only if, it contains isolated strongly connected graphs and at most one strongly connected graph with external quasi-particle lines and if each of the dashed lines connects two parts of the graph which would be disconnected if the dashed lines were not present.

We can now go back and examine the effects of the various possible non-negligible dashed lines. Consider first the one in Fig. 1. According to the rule above, it must appear as shown in Fig. 6(a) or else as in Fig. 6(b) (or perhaps both simultaneously, but then only in a "vacuum" energy term where it is just exactly taken into account by considering only Figs. 6(a) and 6(b); investigation shows that the factors of 2 and 1/2 all occur in exactly the proper places). Figure 6(a),

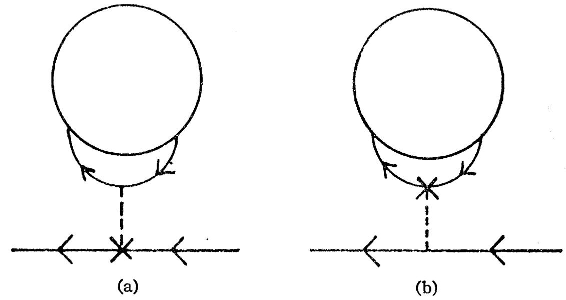


FIG. 6. Non-negligible graphs arising from Fig. 1.

summed over all possibilities for the circle, gives a self-energy which can be represented graphically by Fig. 7. We will give self-energies μ in such a way that the factor corresponding to Fig. 7 is $-2\pi i\mu$. This means that self-energies μ add directly to the unperturbed self-energy $\epsilon(\mathbf{k})$. The self-energy corresponding to Fig. 6(a) is

$$\mu_1(\mathbf{k})=-(w(\mathbf{k})/V)(N-\bar{N}_0), \tag{43}$$

where $\bar{N}_0$ is defined by

$$N-\bar{N}_0=\sum_{\mathbf{k}}{}' \int_{-\infty}^{\infty} d\lambda[G(\mathbf{k},\lambda)-G_0{}^u(\mathbf{k},\lambda)] =\sum_{\mathbf{k}}{}' \int_{\mathrm{up}} d\lambda\, G(\mathbf{k},\lambda), \tag{44}$$

and $G(\mathbf{k},\lambda)$ is the total (perturbed) single quasi-particle Green's function. The notation is appropriate in that it follows from the standard techniques[3] that $\bar{N}_0$ is in fact the expectation value of $\mathfrak{N}_0$ in the perturbed ground

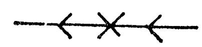


FIG. 7. "Shorthand" graph for Fig. 6(a). A sum over all possibilities for the circle is implied. This is a self-energy term.

state. By $\int_{\mathrm{up}}$ we mean that the integral must be taken around a contour enclosing the upper half λ plane. The second part of (44) follows from the fact that $G_0{}^u$ has poles only in the lower half plane.[4]

Figure 6(b) gives the self-energy

$$\mu_2=-V^{-1}\sum_{\mathbf{k}}{}' \int_{-\infty}^{\infty} d\lambda\, w(\mathbf{k})[G(\mathbf{k},\lambda)-G_0{}^u(\mathbf{k},\lambda)] =-V^{-1}\sum_{\mathbf{k}}{}' \int_{\mathrm{up}} d\lambda\, w(\mathbf{k})G(\mathbf{k},\lambda). \tag{45}$$

[3] A. Klein and R. Prange, Phys. Rev. **112**, 994 (1958).

[4] Another way of obtaining the result that the λ integral may be taken around a contour enclosing the upper half λ plane is to note that the box in Fig. 6(a) represents $\lim \sum_{\mathbf{k}}{}' G(\mathbf{k},\Delta t)$ as $\Delta t\rightarrow 0-$, where $G(\mathbf{k},t)$ is the Fourier transform of $G(\mathbf{k},\lambda)$. The time interval Δt is $t'-t_0$, where t_0 is the time of creation and t' is the time of annihilation of the quasi-particle. We have

$$\lim_{\Delta t\rightarrow 0-} \sum_{\mathbf{k}}{}' G(\mathbf{k},\Delta t)=\lim_{\Delta t\rightarrow 0-} \sum_{\mathbf{k}}{}' \int_{-\infty}^{\infty} d\lambda\, G(\mathbf{k},\lambda)e^{-i\lambda\Delta t}=\sum_{\mathbf{k}}{}' \int_{\mu\mathrm{p}} d\lambda\, G(\mathbf{k},\lambda).$$

Similar analyses are valid for all λ integrations arising from Green's functions whose incoming and outgoing lines have the same time variable.

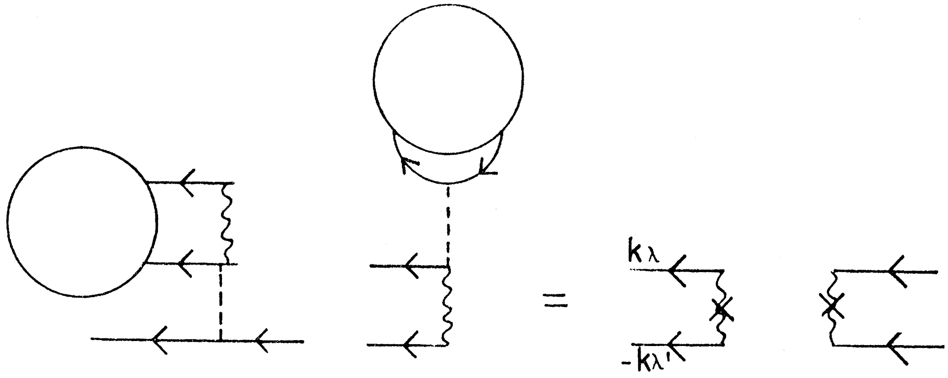


FIG. 8. The self-energy and vertex corrections arising from Fig. 2.

Figure 2(c) gives a self-energy, Fig. 8(a), of

$$\mu_3 = -(2V)^{-1} \sum_{\mathbf{k}}{}' \int d\lambda\, w(\mathbf{k}) G_{+-}(\mathbf{k},\lambda), \qquad (46)$$

where the factor 1/2 is due to the fact that G_{+-} alone would give every graph in μ_3 (except those with a factor of 1/2 due to symmetry) twice. Here $G_{+-}(\mathbf{k},\lambda)$ is the Green's function indicated by Fig. 9, i.e., it

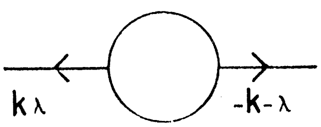


FIG. 9. A graph representing $G_{+-}(\mathbf{k},\lambda)$.

is the sum of all possible connected Feynman graphs of the form shown in Fig. 9. Figure 2(c) also gives the vertex correction shown in Fig. 8(b), with the factor

$$[2\pi i w(\mathbf{k})/V](N-\bar{N}_0)\delta(\lambda+\lambda'). \qquad (47)$$

Clearly, since the Green's function indicated in Fig. 10 is equal, term by term, to that in Fig. 9, it follows that the self-energy arising from Fig. 2(d) is also μ_3. The vertex correction is of the form Fig. 8(c), with the same factor (47).

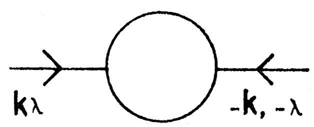


FIG. 10. Another graph representing $G_{+-}(\mathbf{k},\lambda)$.

The dotted lines in Fig. 3 again give both self-energy terms and vertex corrections. The vertex correction due to the dashed lines in Fig. 3(a) comes from the graphs in which all the lines arising from the factors $\mathfrak{N}_{\mathbf{k}}$ are closed as shown in Fig. 11(a). The self-energy is due to graphs of the type shown in Fig. 11(b). For the vertex correction we find

$$(-2\pi i w(\mathbf{k}_1)/V) \sum_{l=1}^{\infty} \left[\frac{1}{l!}\left(\frac{d}{dx}\right)^l (N-x)^{1/2}\right]_{x=0} (N-\bar{N}_0)^l$$
$$= -2\pi i \bar{N}_0^{1/2} w(\mathbf{k}_1)/V + 2\pi i N^{1/2} w(\mathbf{k}_1)/V. \qquad (48)$$

Figure 11(b) gives the self-energy

$$\mu_4 = V^{-1}\left\{ \sum_{l=1}^{\infty} \left[\frac{1}{(l-1)!}\left(\frac{d}{dx}\right)^l (N-x)^{1/2}\right]_{x=0} (N-\bar{N}_0)^{l-1}\right\}$$
$$\times \sum_{\mathbf{k}_1\mathbf{k}_2}{}' \int_{\text{up}} d\lambda_1 \int d\lambda_2\, w(\mathbf{k}_1) G_a(\mathbf{k}_1\mathbf{k}_2,\lambda_1\lambda_2)$$
$$= -(2\bar{N}_0^{1/2}V)^{-1} \sum_{\mathbf{k}_1\mathbf{k}_2} \int_{\text{up}} d\lambda_1 \int d\lambda_2$$
$$\times w(\mathbf{k}_1) G_a(\mathbf{k}_1\mathbf{k}_2,\lambda_1\lambda_2), \qquad (49)$$

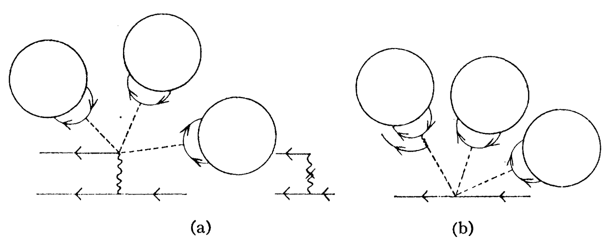
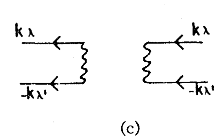


FIG. 11. The self-energy and vertex corrections arising from Fig. 3.



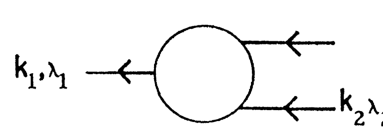
where G_a is the Green's function shown in Fig. 12. Figure 3(b) gives a self-energy μ_4 again and a vertex correction indicated in Fig. 11(c) with a factor given by (48). Thus, the effect of all the self-energy terms is to change the denominator of the single quasi-particle Green's function from

$$\lambda - k^2 - Nw(\mathbf{k})/V,$$

to

$$\lambda - k^2 - Nw(\mathbf{k})/V - \mu_1(\mathbf{k}) - \mu_2 - 2\mu_3 - 2\mu_4,$$

FIG. 12. A graph representing G_a.



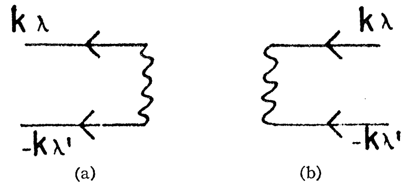


FIG. 13. Vertices remaining in the renormalized problem which involve the creation of two quasi-particles or the annihilation of two quasi-particles.

which is, with (43),

$$\lambda-k^2-\bar{N}_0w(\mathbf{k})/V-\mu, \tag{50}$$

where

$$\mu=\mu_2+2\mu_3+2\mu_4 \tag{51}$$

is independent of $\mathbf{k}$.

The vertices that result when similar vertices are combined are shown in Figs. 13, 14, and 4. The factor

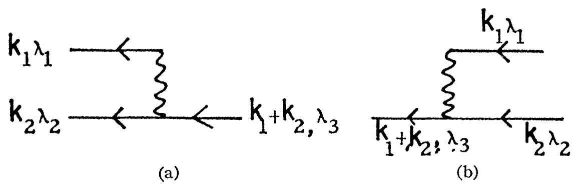


FIG. 14. Vertices remaining in the renormalized problem which involve the creation of two quasi-particles and the annihilation of one quasi-particle or the creation of one quasi-particle and the annihilation of two quasi-particles.

associated with Fig. 13 is the sum of (39) and (47 and is

$$[-2\pi i\bar{N}_0w(\mathbf{k})/V]\delta(\lambda+\lambda'). \tag{52}$$

The factor associated with each of the vertices in Fig. 14 is the sum of (48) and the term with $m=0$ in (41):

$$[-2\pi i\bar{N}_0{}^{1/2}w(\mathbf{k}_1)/V]\delta(\lambda_1+\lambda_2-\lambda_3). \tag{53}$$

The factor associated with Fig. 4 is still (42).

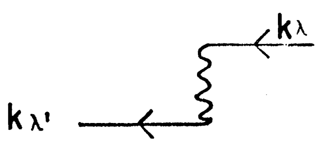


FIG. 15. Vertex introduced in the renormalized problem.

In order to simplify the future treatment, we define our new single-quasi-particle Green's function $G_0(\mathbf{k},\lambda)$ to be

$$G_0(\mathbf{k},\lambda)=(i/2\pi)(\lambda-k^2-\mu+i0)^{-1}, \tag{54}$$

and incorporate the term $\bar{N}_0w(\mathbf{k})/V$ in (50) as the vertex shown in Fig. 15 with factor

$$[-2\pi i\bar{N}_0w(\mathbf{k})/V]\delta(\lambda-\lambda'). \tag{55}$$

We are then above to restate the diagram rules in a very simple form. There are directed quasi-particle lines, Fig. 16(a), with corresponding factors G_0 (54)

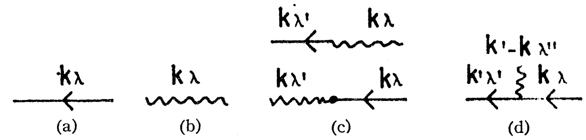


FIG. 16. Elements of diagrams in the renormalized problem.

with appropriate wave number and frequency. There are wavy interaction lines, Fig. 16(b), with associated factors $D_0(\mathbf{k},\lambda)$:

$$D_0(\mathbf{k},\lambda)=-2\pi iw(\mathbf{k})/V, \tag{56}$$

and two types of vertices as shown in Figs. 16(c) and 16(d). The "incomplete" vertices in Fig. 16(c) have a factor $\bar{N}_0{}^{1/2}$; the "complete" vertex in Fig. 16(d) has a factor of unity associated with it. In addition, there are just the few combinatorial factors mentioned directly

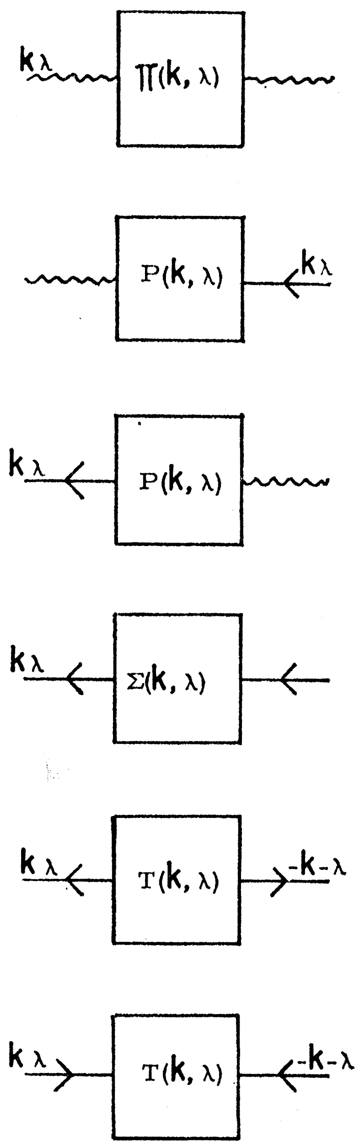


FIG. 17. Graphs representing the proper parts.

after (42). Wave number and frequency are conserved at each vertex.

The previous paragraph defines what we shall call the renormalized problem. What we have done up to this point is to prove that the renormalized problem is equivalent to the problem specified by the original Hamiltonian (1) with the subsidiary condition that the total number of particles be N, the proof being valid if N and the volume of the system are large enough. In the renormalized problem μ is given by (51), (45), (46), and (49), while $\bar{N}_0$ is given by (44). The equations for μ and $\bar{N}_0$ are, therefore, implicit equations; this is the price that is paid for the diagrammatic simplicity of the renormalized problem.

The interpretation of the renormalized problem as involving "interactions" (the wavy lines) which propagate with infinite velocity, corresponding to D_0 being independent of frequency, is obvious. In this picture $\bar{N}_0^{1/2}$ is just the coupling constant for transformation of an interaction into a quasi-particle or vice versa. Corrections for finite velocity of propagation of the interaction would then introduce a frequency dependence of D_0.

VI. PROPER PARTS AND GREEN'S FUNCTIONS

Because of the simplicity of the rules for Feynman diagrams in the renormalized problem, it is advantageous to introduce the proper parts shown in Fig. 17, where each of the proper parts represents all possible connected graphs which cannot be disconnected by cutting a single line of either type. Since k^2 and $w(\mathbf{k})$ are even in $\mathbf{k}$, Π, P, Σ and T are even in $\mathbf{k}$. Also, it is clear that Π and T are even in λ.

We introduce the Green's functions shown in Fig. 18. All are even in $\mathbf{k}$, and G_{+-} and D are also even in λ. As

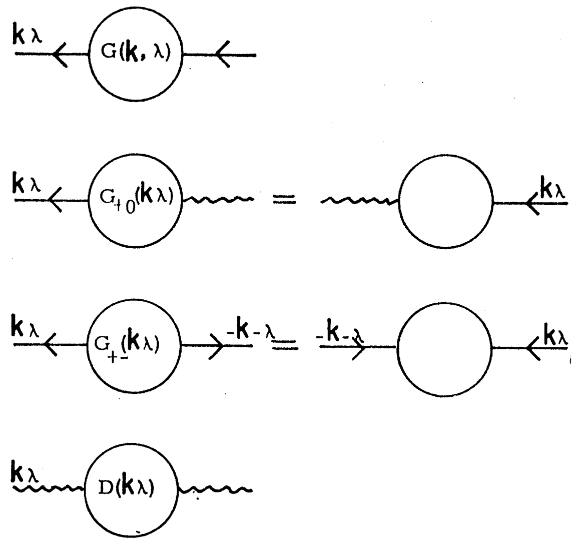


FIG. 18. Graphs representing the Green's functions.

usual, the Green's functions include the factors for the end lines while the proper parts do not.

Figure 19 shows diagrammatically the equations for finding some of the G's in terms of the proper parts; the equations themselves are

$$G=G_0+G\Sigma G_0+G_{+0}\mathrm{P}G+G_{+-}\mathrm{T}G_0,$$
$$G_{+0}=G\mathrm{P}D_0+G_{+0}\Pi D_0+G_{+-}\mathrm{P}^{(-)}D_0, \tag{57}$$

and

$$G_{+-}=G\mathrm{T}G_0^{(-)}+G_{+0}\mathrm{P}^{(-)}G_0^{(-)}+G_{+-}\Sigma^{(-)}G_0^{(-)},$$

where a $(-)$ superscript indicates that the arguments of the corresponding function are $-\mathbf{k}$ and $-\lambda$. All

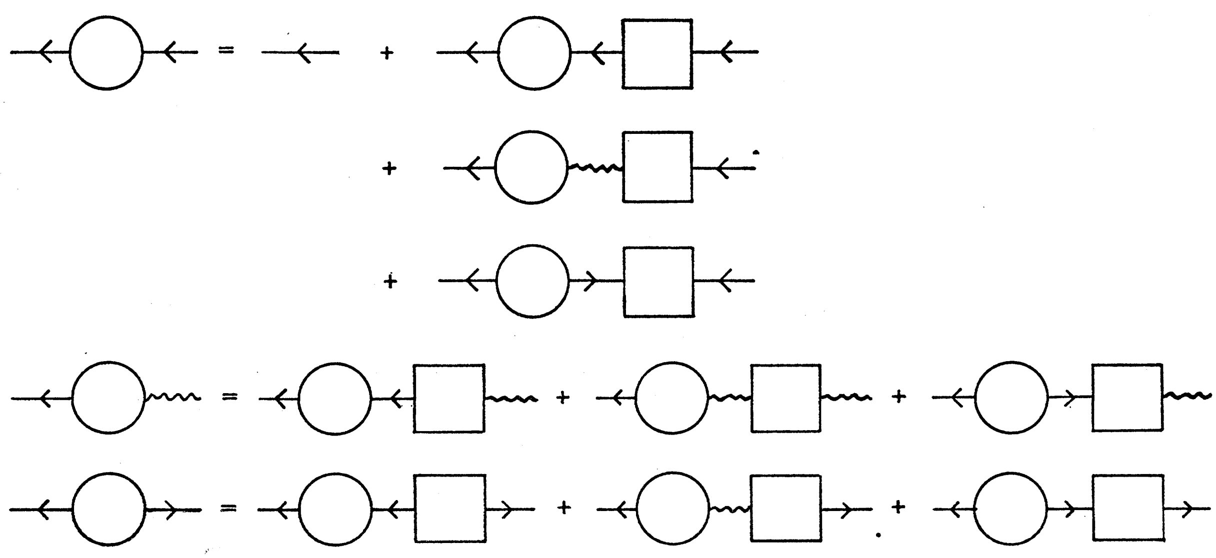


FIG. 19. Graphs illustrating Eq. (57).

other arguments are $\mathbf{k}$, λ. The solution of (57) is

$$G=\Delta^{-1}\begin{vmatrix} D_0^{-1}-\Pi & -\mathrm{P}^{(-)} \\ -\mathrm{P}^{(-)} & G_0^{(-)-1}-\Sigma^{(-)} \end{vmatrix},$$

$$G_{+0}=\Delta^{-1}\begin{vmatrix} -\mathrm{P}^{(-)} & G_0^{(-)-1}-\Sigma^{(-)} \\ -\mathrm{P} & -\mathrm{T} \end{vmatrix}, \tag{58}$$

$$G_{+-}=\Delta^{-1}\begin{vmatrix} -\mathrm{P} & D_0^{-1}-\Pi \\ -\mathrm{T} & -\mathrm{P}^{(-)} \end{vmatrix}$$

where

$$\Delta=\begin{vmatrix} G_0^{-1}-\Sigma & -\mathrm{P} & -\mathrm{T} \\ -\mathrm{P} & D_0^{-1}-\pi & -\mathrm{P}^{(-)} \\ -\mathrm{T} & -\mathrm{P}^{(-)} & G_0^{(-)-1}-\Sigma^{(-)} \end{vmatrix}, \tag{59}$$

Δ is even in $\mathbf{k}$ and λ. By a similar procedure we find

$$D=\Delta^{-1}\begin{vmatrix} G_0^{-1}-\Sigma & -\mathrm{T} \\ -\mathrm{T} & G_0^{(-)-1}-\Sigma^{(-)} \end{vmatrix}. \tag{60}$$

As usual, the poles of these Green's functions are related to excitation energies of the system. In the present case, we have Green's functions for quasi-particles which involve no change in the number of particles. The zeros of Δ are, therefore, excitation energies of the system.

VII. GROUND-STATE ENERGY, μ, AND N_0

In the renormalized problem, every isolated graph is of order VT, where V is as before and the T comes from the fact that such a graph has an extra frequency delta function which is to be interpreted as

$$\delta(0)=\lim_{T\to\infty}(2\pi)^{-1}\int_{-T/2}^{T/2}e^{i0t}dt=\lim_{T\to\infty}\frac{T}{2\pi}. \tag{61}$$

Let $LT/2\pi$ be the sum of all the different diagrams each consisting of a single isolated graph. Then the factorial factors for diagrams with identical isolated graphs ensure that the sum of all diagrams with no external lines is $\exp(LT/2\pi)$. This is related to the ground-state energy E_0 by

$$e^{-iE_0T}=e^{LT/2\pi}, \tag{62}$$

or

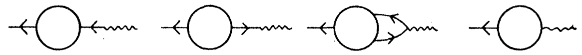
$$E_0=iL/2\pi.$$

Actually, of course, we must still add E_1 so that

$$E_0=E_1+iL/2\pi. \tag{63}$$

Consider now $(\partial E_0/\partial\mu)_{\bar{N}_0}$, where we mean that μ occurs in each term only in G_0 (54) and $\bar{N}_0$ only at the incomplete vertices, Fig. 16(c). Let L_i be a graph of L. Then $(i/2\pi)(\partial L_i/\partial\mu)_{\bar{N}_0}$ is L_i, with each quasi-particle Green's function doubled up. From Fig. 6(a) it is clear that this is just what is needed for $N-\bar{N}_0$:

$$N-\bar{N}_0=(i/2\pi)(\partial L/\partial\mu)_{\bar{N}_0}=(\partial E_0/\partial\mu)_{\bar{N}_0}. \tag{64}$$



(a) (b) (c) (d)

FIG. 20 Graphs illustrating the simplification of the expression for μ.

The factor 1/2 in certain E_0 graphs just takes care of the fact that some terms in $\bar{N}_0$ would seem to occur more than once in (64).

We now consider μ. Diagrammatically, we can represent μ_2, $2\mu_3$ and $2\mu_4$ as in Fig. 20(a), (b), and (c); each figure must be multiplied by $(2\pi i\bar{N}_0^{1/2})^{-1}$ and integrated over the $\mathbf{k}$ and λ which appear. Clearly the sum is just Fig. 20(d), and, therefore,

$$\mu=(2\pi i\bar{N}_0^{1/2})^{-1}\sum_{\mathbf{k}}{}'\int_{\mathrm{up}}d\lambda G_{+0}(\mathbf{k},\lambda). \tag{65}$$

It is also evident that if G_{+0}^i is a term of G_{+0}, then $\bar{N}_0^{1/2}\sum_{\mathbf{k}}{}'\int_{\mathrm{up}}G_{+0}^i d\lambda$ is a term in E_0. However, each term L_i in E_0 corresponds to just as many terms in $\bar{N}_0^{1/2}\sum_{\mathbf{k}}{}'\int_{\mathrm{up}}G_{+0}^i d\lambda$ as there are incomplete vertices of a particular type in L_i. Since the numbers of incomplete vertices of the two types are equal, the number of either is just the exponent of $\bar{N}_0$:

$$\bar{N}_0^{1/2}\sum_{\mathbf{k}}{}'\int_{\mathrm{up}}G_{+0}d\lambda=\bar{N}_0\frac{\partial L}{\partial\bar{N}_0} \tag{66}$$

and, therefore,

$$\mu=(1/2\pi i)(\partial L/\partial\bar{N}_0)=-\partial E_0/\partial\bar{N}_0. \tag{67}$$

Equation (67) is almost Belyaev's rule for determining μ. However, Belyaev has $\mu=-\partial E_0/\partial N$, while the above derivation shows that $\bar{N}_0$ is correct.

VIII. NONEXISTENCE OF ENERGY GAP

We can also describe the limiting values of the proper parts in terms of derivatives of E_0. For example, $\Sigma(0,0)$ with its external lines removed is just like $L/\bar{N}_0$, except that each term in E_0 gives a^2 contributions to $\Sigma(0,0)$, where a is the number of incomplete vertices of a given type.

Therefore,

$$\begin{aligned}\Sigma(0,0)&=(\partial/\partial\bar{N}_0)\bar{N}_0(\partial L/\partial\bar{N}_0)\\&=(\partial L/\partial\bar{N}_0)+\bar{N}_0(\partial^2L/\partial\bar{N}_0^2).\end{aligned} \tag{68}$$

$\mathrm{P}(0,0)$ without its external lines is $L/\bar{N}_0^{1/2}$ with a doubled quasi-particle Green's function (except for $\bar{N}_0^{1/2}$ which appears in lowest order):

$$\mathrm{P}(0,0)=(i/2\pi)(\partial/\partial\mu)\bar{N}_0^{1/2}(\partial L/\partial\bar{N}_0)+\bar{N}_0^{1/2}. \tag{69}$$

Similarly

$$T(0,0)=\bar{N}_0(\partial^2L/\partial\bar{N}_0^2), \tag{70}$$

$$\Pi(0,0)=(i/2\pi)(\partial^2L/\partial\mu^2). \tag{71}$$

Therefore, we have

$$\Delta(0,0)=\begin{vmatrix} 2\pi i\mu-\dfrac{\partial L}{\partial\bar{N}_0}-\bar{N}_0\dfrac{\partial^2 L}{\partial\bar{N}_0{}^2} & -\bar{N}_0{}^{1/2}-\dfrac{i\bar{N}_0{}^{1/2}}{2\pi}\dfrac{\partial^2 L}{\partial\mu\partial\bar{N}_0} & -\bar{N}_0\dfrac{\partial^2 L}{\partial\bar{N}_0{}^2} \\ -\bar{N}_0{}^{1/2}-\dfrac{i\bar{N}_0{}^{1/2}}{2\pi}\dfrac{\partial^2 L}{\partial\mu\partial\bar{N}_0} & \dfrac{iV}{2\pi w(0)}-\left(\dfrac{i}{2\pi}\right)^2\dfrac{\partial^2 L}{\partial\mu^2} & -\bar{N}_0{}^{1/2}-\dfrac{i\bar{N}_0{}^{1/2}}{2\pi}\dfrac{\partial^2 L}{\partial\mu\partial\bar{N}_0} \\ -\bar{N}_0\dfrac{\partial^2 L}{\partial\bar{N}_0{}^2} & -\bar{N}_0{}^{1/2}-\dfrac{i\bar{N}_0{}^{1/2}}{2\pi}\dfrac{\partial^2 L}{\partial\mu\partial\bar{N}_0} & 2\pi i\mu-\dfrac{\partial L}{\partial\bar{N}_0}-\bar{N}_0\dfrac{\partial^2 L}{\partial\bar{N}_0{}^2} \end{vmatrix}=0, \quad (72)$$

since the first and third rows are equal when (67) is used.

Thus, $\lambda=0$ is a zero of $\Delta(0,\lambda)$ and, therefore, the lowest excitation of the system has zero energy for $\mathbf{k}\to 0$. This is the result that was first proved by Hugenholtz and Pines.[5]

IX. LOWEST ORDER

The incomplete vertices are more important than the complete ones by a factor $\bar{N}_0{}^{1/2}$. We, therefore, define the lowest order to be that which considers the incomplete vertices only to all orders. This is very simply done in terms of the proper parts we have introduced. With the label 1 designating this lowest order, we have

$$\Pi_1=\Sigma_1=\mathrm{T}_1=0,\quad \mathrm{P}_1(k,\lambda)=\bar{N}_0{}^{1/2}. \quad (73)$$

Substitution into (59), then, gives

$$\Delta_1=(2\pi iV/w(\mathbf{k}))\{\lambda^2-(k^2+\mu)^2 -[2\bar{N}_0w(\mathbf{k})/V](k^2+\mu)\}. \quad (74)$$

From the preceding proof of the nonexistence of an energy gap, in particular from (68)–(71), it follows that we must take no contributions to E_0 in this order

$$E_{0(1)}=E_1, \quad (75)$$

and, therefore, from (64) and (67),

$$\mu_1=(N-\bar{N}_0)_1=0, \quad (76)$$

and

$$\Delta_1=[2\pi iV/w(\mathbf{k})\{\lambda^2-k^2[k^2+2Nw(\mathbf{k})/V]\}, \quad (77)$$

with a spectrum

$$\lambda_1=k[k^2+2\rho w(\mathbf{k})]^{1/2}, \quad (78)$$

which is the one that has often been given for this system.

Note that it is helpful, as in (75), to use the procedures of the previous section as a guide for determining what selection of graphs will be consistent with the nonexistence of the gap. Of course, it is possible to find a gap by terminating the perturbation series for E_0 and for the proper parts at points which are not consistent with the proof of the previous section. The proof itself depends, clearly, on convergence of the perturbation series.

ACKNOWLEDGMENTS

One of the authors (M. B.) would like to thank Professor D. R. Yennie and Dr. H. Suura for helpful discussions.

[5] N. M. Hugenholtz and D. Pines, Phys. Rev. **116**, 489 (1959).